A Simple, Robust Identification Approach for First-Price Auctions

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Abstract

This paper proposes a new approach to the identification of first-price auctions that is robust to overbidding, but at the same time remains contiguous with the canonical point-identification approach of Guerre, Perrigne, and Vuong (2000) (GPV) and its simple estimators. We show that a weak identifying restriction allows us to reinterpret the GPV estimates as a bound. We demonstrate that the identifying restriction holds in a set of commonly used auction models that can generate overbidding and is satisfied in the bid data from a laboratory experiment. We illustrate the approach in applications to laboratory data and field data. We recommend that practitioners continue to follow the GPV approach, but interpret the estimates as a bound in applications where they are concerned about overbidding.

1 Introduction

Point identification of structural models typically requires strong assumptions. This motivates the development of partial identification approaches that are robust to violations of these assumptions. However, partial identification approaches are often complicated, require cumbersome estimation procedures, and the estimates can be difficult to interpret. Therefore, practitioners often fall back on point identification approaches even if they are worried about the strong underlying assumptions.

Identification of first-price auctions is a good example of a structural model that relies on strong assumptions. The canonical identification approach developed by Guerre, Perrigne, and Vuong (2000) (GPV) assumes that the bidders play the risk-neutral Bayesian Nash Equilibrium (RNBNE), which is a demanding equilibrium concept. There is indeed empirical evidence from numerous laboratory experiments with independent private values showing a general tendency towards overbidding compared to the RNBNE prediction. In the theoretical literature various alternatives

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to the RNBNE model have been considered to accommodate these findings in the lab, for example by introducing risk-aversion or ambiguity aversion.[1] In practice, it is however unclear which of the alternative models should be used. Grundl and Zhu (2023a) therefore proposed a partial identification approach assuming that bidders tend to overbid compared to the RNBNE. This approach accommodates a range of these alternative models, but estimation is more difficult than the point estimator of GPV, which is based on a closed-form expression and does not involve optimization. Moreover, to obtain tight bounds sometimes requires further restrictions on the shape of the underlying valuation distribution, which further complicates the estimation.

This paper tries to combine the advantages of GPV and Grundl and Zhu (2023a). We obtain a weak identifying restriction under which the GPV estimates can be reinterpreted as a sharp bound for the valuation distribution even if the bidders violate the RNBNE assumption. Under additional restrictions on counterfactual bidding behavior the bounds for the valuation distribution can be translated into bounds for the counterfactual profit with a binding reserve price. We show that the identifying restriction is stronger than the one used by Grundl and Zhu (2023a), but is still consistent with a range of overbidding models and holds in the experimental bid data by Dyer, Kagel, and Levin (1989). The resulting bounds are tighter than those proposed by Grundl and Zhu (2023a) both in theory and in empirical applications.

Our starting point is the observation that not all violations of the RNBNE prediction are equally common and instead there is a systematic pattern such that bidders overbid compared to RNBNE bidding.[2] This observation suggests that the GPV approach may still yield a valid bound under the assumption that the bidders overbid compared to the RNBNE bid strategy.

It turns out, however, that overbidding compared to RNBNE is not sufficient to interpret GPV as a bound. To understand this recall that GPV is based on the insight that if the bidders bid according to the RNBNE, then the RNBNE strategy is identical to the risk neutral best response to the bid distribution (RNBR). Therefore, the econometrician can estimate the RNBNE bid function by estimating the best response to the observed bid distribution. If the bidders deviate from RNBNE bidding however, the RNBNE and RNBR strategies are in general no longer identical. This is because deviations from RNBNE bidding change the bid distribution and therefore the RNBR strategy. This gap between the RNBNE and RNBR strategies explains why assuming that the bidders overbid compared to the RNBNE is not sufficient to interpret the GPV estimator as a bound.[3]

[1] For some of these models identification results have been obtained in the structural literature by imposing additional identifying restrictions. For example, for models with risk aversion (Perrigne and Vuong (2007), Lu and Perrigne (2008), Guerre, Perrigne, and Vuong (2009), Campo, Guerre, Perrigne, and Vuong (2011), Campo (2012), Gentry, Li, and Lu (2015), Kong (2017)), ambiguity aversion (Aryal, Grundl, Kim, and Zhu (2018)), and the level-k model (An (2017), Gillen (2009)). It is difficult however to distinguish among these models with bid data alone (see Theorems 1 and 2 on non-testability in Grundl and Zhu (2023a)), and not even the experimental literature where valuations are observed as well, has settled on a single alternative model. Moreover the mechanisms in all of these models could be contributing to the observed deviations from RNBNE bidding at the same time. Therefore, we propose a bounds approach that is consistent with various alternative bidding models.

[2] For references to the large experimental literature on overbidding compared to RNBNE see Grundl and Zhu (2023a) and the surveys by Kagel (1995) and Kagel and Levin (2010).

[3] Assuming overbidding compared to RNBNE leads to a different bound (Grundl and Zhu (2023a)), but this bound

Instead, to interpret the GPV estimates as a bound we need to consider an alternative notion of overbidding, namely that the bidders overbid compared to the RNBR strategy (RNBR overbidding). This notion of overbidding has not received much attention in the experimental literature, because the experimental literature is primarily concerned with deviations from the predictions of economic theory not with identifying assumptions in the structural auction literature. We show that under the assumption of RNBR overbidding the GPV estimator can indeed be interpreted as a lower bound for the valuation distribution. It is not necessary to assume that all bidders overbid compared to RNBR, but only that there is a general tendency towards overbidding such that there is a first-order stochastic dominance ranking between the actual bid distribution and the bid distribution implied by RNBR bidding ("overbidding in distribution").

We further show that under additional restrictions the counterfactual profit under a binding reserve price constructed using the GPV estimator is a sharp upper bound for the true profit. Therefore, both the GPV value distribution and the profit function can be interpreted as bounds.

While RNBR overbidding in distribution is a convenient identifying assumption it is not clear whether it is also a plausible restriction. Indeed we show that under symmetry RNBR overbidding is stronger than RNBNE overbidding. Therefore, the large body of evidence showing RNBNE overbidding in experiments is not sufficient to support the RNBR overbidding assumption. We show empirically, using the experimental data by Dyer, Kagel, and Levin (1989), that the bidders overbid compared to RNBR in distribution. Moreover, we show theoretically that RNBR overbidding arises naturally in a wide range of models that can generate RNBNE overbidding.

In an empirical application to laboratory data we compare the bounds arising from the assumption of RNBNE overbidding in distribution developed by Grundl and Zhu (2023a) and from RNBR overbidding in distribution developed in this paper. The RNBR bounds contain the true valuation distribution quantile function and are substantially tighter than the RNBNE bounds. Intuitively, the bounds are obtained by taking the lower and upper envelopes of sets of distributions that satisfy rationality and the corresponding overbidding restrictions. Because RNBR overbidding in distribution is a stronger condition than RNBNE overbidding in distribution, the set of valuation distributions that satisfy the former is smaller than the set of distributions that satisfy the latter, leading to tighter bounds.

We also apply the approach to field data from USFS timber auctions. In this application we focus on tightening the bounds using variation in the number of bidders and on interpreting the estimated bounds for the seller's profit. Perhaps the main insight is that bounds for the profit widen asymmetrically as the reserve price increases. Without a binding reserve price the upper and lower profit bounds coincide, because this case is observed directly in the data. As the reserve price increases, the lower profit bound decreases substantially, while the upper bound first increases modestly and then decreases modestly. The seller may therefore prefer not to set a binding reserve price, which maximizes the lower profit bounds, rather than to maximize the upper profit bound, which could lead to a modest gain in profits, but could also lead to a substantial reduction of profits.

is less informative and more difficult to estimate than GPV. We compare both identification approaches in this paper.

For practitioners we recommend that they keep using the GPV estimator, but interpret the estimates in two different ways if they are concerned about overbidding. First, interpret GPV as a point estimate that yields a single optimal reserve price. Second, interpret GPV as a robust bound, which yields a max-max reserve price or a range of optimal reserve prices if coupled with the bound from the other side. The nice feature is that in this context practitioners do not have to decide at the outset whether to pursue a point identification approach or a more robust partial identification approach. Instead they can simply interpret the same estimate in two different ways.

One important practical question is when the robust approach proposed in this paper can and should be employed in applications to field data. Unlike in laboratory experiments valuations are not observed in field data so the detection of overbidding is less straightforward. There are however findings in the literature that point towards the importance of overbidding in at least some applications to field data.[4] While our approach is robust to overbidding, it is not robust to underbidding, which can occur if bidders collude or if they have biased beliefs. While a rigorous test of overbidding is beyond the scope of this paper one approach is to estimate valuation quantiles under the assumption of RNBNE bidding for different numbers of bidders. We show theoretically that if the estimated valuation quantiles under the RNBNE assumption are shifted to the left as the number of bidders increases this could be indicative of overbidding.[5]

This paper is most closely related to Haile and Tamer (2003) and to Grundl and Zhu (2023a). Haile and Tamer (2003) show how to construct simple and robust bounds for English auctions with an incomplete model of bidding behavior. Grundl and Zhu (2023a) propose robust bounds in first-price auctions based on the assumption that bidders overbid compared to the RNBNE, but the bounds require solving a high dimensional linear programming problem, which can be difficult to compute. The alternative bounds proposed in this paper can be estimated using standard estimation methods for the GPV inverse in the literature. The resulting bounds are not only easier to compute, but also more informative. This paper can therefore be viewed as providing an analogue to Haile and Tamer (2003) for first-price auctions. Relatedly, Armstrong (2013) shows that GPV leads to a valid upper bound on the mean of the valuations if there is auction-level unobserved heterogeneity. Tang (2011) studies the common value setup and derives bounds for the seller's profit function under binding reserve prices.

What are the main differences between the assumptions in Grundl and Zhu (2023a) and in this paper? First, the identifying assumption used in this paper is slightly stronger than in Grundl and Zhu (2023a). While Grundl and Zhu (2023a) assumes overbidding compared to the RNBNE bidding strategy we assume overbidding compared to the RNBR strategy. Under symmetry RNBR overbidding implies RNBNE overbidding (Proposition 1 and Corollary 1) and is therefore stronger. We show however that RNBR overbidding is only a slightly stronger assumption in the sense that

[4] For instance, Lu and Perrigne (2008), Campo, Guerre, Perrigne, and Vuong (2011), Campo (2012) and Kong (2017) find that bidders are risk averse. Grundl and Zhu (2019) and Gimenes and Guerre (2022) however find no evidence of risk aversion. Aryal, Grundl, Kim, and Zhu (2018) find that bidders overbid due to ambiguity aversion. Overbidding in M&A contests has been documented by Burkart (1995) and de Bodt, Cousin, and Roll (2018).

[5] See Proposition 8 in Section 4.

most commonly used auction models that are consistent with RNBNE overbidding are also consistent with RNBR overbidding (Proposition 2). Moreover, we show empirically that RNBR overbidding in distribution holds in the experimental data by Dyer, Kagel, and Levin (1989). Second, the assumption imposed for the counterfactual (Assumption 5), i.e. the restriction of bidding with a binding reserve price, is also stronger than the corresponding Assumption 6 in Grundl and Zhu (2023a). This mirrors the stronger identifying assumption: While Assumption 6 in Grundl and Zhu (2023a) only implies overbidding compared to RNBNE, Assumption 5 in this paper implies overbidding compared to RNBR.[6]

The remainder of the paper is organized as follows. Section 2 lays out the identification problem and introduces the key identifying assumption. Section 3 studies the plausibility of the key identifying assumption of overbidding compared to RNBR, and compares it to overbidding compared to RNBNE. Section 4 introduces the main identification results that allow us to reinterpret the GPV estimates as a bound. Section 5 provides estimators for the derived bounds. Section 6 presents applications to data from a laboratory experiment and to field data from timber auctions. Section 7 considers two extensions where bidders have different types differing in bidding strategies. In one extension, bidders are unaware of the heterogeneity in types and in the other, bidder types are common knowledge. Section 8 concludes. Proofs that are not given in the main text are collected in the Online Appendix.

2 Identification Problem and Identifying Assumptions

Consider a symmetric independent private-value environment. There are $n \geq 2$ bidders participating in a first-price auction for an indivisible good without a reserve price. Each of them simultaneously submits a sealed bid. The bidder with the highest bid wins the good and pays the bid. Bidders' valuations t for the good are independently drawn from the distribution F_n with density f_n. The bidders use the common bidding strategy s_n. The bid distribution is denoted by G_n with density g_n. We assume that g_n is supported on a bounded interval, bounded away from 0 and continuous almost everywhere on its support but may have a finite number of downward jumps.[7] Similarly, f_n is bounded away from 0 on its support. Without loss of generality, we assume that the support of f_n has a lower bound of 0. Our goal is to identify the valuation distribution F_n from the bid distribution G_n.

Let σ_n be the bidding strategy in the symmetric Risk Neutral Bayesian Nash Equilibrium

[6]Besides these main differences in assumptions there are also some smaller differences. Grundl and Zhu (2023a) assumes that the valuation distribution is unimodal to tighten the bounds, whereas the bounds in this paper are fairly tight without such an assumption. Assumption 1 in this paper ensures that the RNBR strategy is characterized by the first-order condition and monotone. This assumption is not needed in Grundl and Zhu (2023a), because it does not use the RNBR strategy.

[7]We allow for heterogeneous bidding strategies so the bid distribution is a mixture of different bid distributions. We assume that this bidder heterogeneity is not known to the rival bidders or to the econometrician, but in Section 7 we consider an extension where it is known. To accommodate finite mixtures of different bid distributions, g_n is allowed to have a finite number of downward jumps.

(RNBNE). Then for any t,

$$\sigma_n(t) = \arg\max_b (t-b) F_n\left(\sigma_n^{-1}(b)\right)^{n-1}.$$

Intuitively, given all other bidders play σ_n, a bidder with a valuation t chooses b to maximize her expected payoff under risk neutrality, trading off the profit from winning $t-b$ and the probability of winning $F_n\left(\sigma_n^{-1}(b)\right)^{n-1}$. Symmetry requires the optimal strategy coincides with σ_n. One can derive the following closed form expression for σ_n:

$$\sigma_n(t) = t - \int_0^t \left[\frac{F_n(x)}{F_n(t)}\right]^{n-1} dx.$$

Let ρ_n be the bidding strategy that is the Risk Neutral Best Response (RNBR) to the bid distribution:

$$\rho_n(t) = \arg\max_b (t-b)\,\mathbb{P}(b \text{ is the largest bid}) = \arg\max_b (t-b) G_n(b)^{n-1}.$$

We will maintain the following regularity assumption on G_n throughout the paper.

Assumption 1 (Monotonicity). *$b+G_n(b)/[g_n(b)(n-1)]$ is strictly increasing in b on the support of g_n.*

Assumption 1 ensures that $\rho_n(t)$ is strictly increasing in t and is fully characterized by its first-order condition. This assumption is not needed in Grundl and Zhu (2023a) because they rely on the RNBNE strategy instead of the RNBR strategy.

It is standard practice in the empirical auction literature to assume that the bidders play the symmetric Risk Neutral Bayesian Nash Equilibrium (RNBNE) so $s_n = \sigma_n$. This assumption implies that the bidders play ρ_n. Using this insight, Guerre, Perrigne, and Vuong (2000) show that one can identify the inverse bidding function, which is often called the GPV inverse function,

$$s_n^{-1}(b) = \sigma_n^{-1}(b) = \rho_n^{-1}(b) = b + \frac{1}{n-1}\frac{G_n(b)}{g_n(b)}.$$

This directly leads to identification of the valuation distribution as $F_n(t) = G_n(\rho_n(t))$.

In lab experiments and to a lesser extent in field data, however, there is evidence that bidders overbid compared with the RNBNE strategy. With overbidding compared to RNBNE, $\rho_n^{-1}(b)$ is no longer the correct inverse bidding function. A natural question is whether $G_n(\rho_n(t))$ is a valid lower bound for $F_n(t)$. As is shown in the next section, this is however not the case because $s_n(t) \geq \sigma_n(t)$ does not imply that $s_n(t) \geq \rho_n(t)$. For $G_n(\rho_n(t))$ to be a valid lower bound for $F_n(t)$, one sufficient condition is the following.

Assumption 2 (RNBR Overbidding). *Bidders overbid compared to the RNBR bidding strategy: $s_n(\cdot) \geq \rho_n(\cdot)$.*

Assumption 2 requires that every bidder overbids weakly compared to RNBR. This may be a strong assumption in many applications. Indeed, as we discuss in more detail later, experimental data from Dyer, Kagel, and Levin (1989) show that a minority of bidders bid below the RNBR. It turns out that there is a weaker stochastic version of Assumption 2, which is sufficient and necessary for interpreting the GPV estimator as a bound.

Assumption 3 (RNBR Overbidding in Distribution)**.** *Bidders overbid compared to RNBR in distribution:* $F_n\left(\rho_n^{-1}(\cdot)\right) \geq G_n(\cdot)$.

This assumption requires that there is a first-order stochastic dominance ranking between the actual bid distribution and the bid distribution implied by RNBR bidding. It is satisfied if there is a general tendency to bid above the RNBR but some bidders can bid below the RNBR. It fails however if underbidding is too prevalent. Section 3.1 shows both theoretically and using a numerical example that Assumption 3 holds if the fraction of bidders who underbid is positive but sufficiently small. Therefore, Assumption 3 is weaker than Assumption 2 and it allows for heterogeneity in bidders' bidding strategies. Interestingly, one can show that if G_n can be rationalized by some F_n under Assumption 3, it can also be rationalized by F_n and a symmetric bid function $s_n(\cdot)$ that satisfies Assumption 2. To see this, just define $s_n(t) = G_n^{-1}(F_n(t))$ and by Assumption 3, $s_n(t) \geq \rho_n(t)$. In other words, if only G_n is observed, RNBR overbidding and RNBR overbidding in distribution are observationally equivalent. In Section 3.2 we test Assumption 3 in the experimental data by Dyer, Kagel, and Levin (1989) and fail to reject it.[8]

We also assume that bidders bid less than their valuations. We refer to this assumption as "rationality".

Assumption 4 (Rationality)**.** *Bidders do not bid above their valuation.*

3 RNBR Overbidding: Theory and Data

Before moving to the identification results, we would like to discuss the plausibility of the key identifying assumption, RNBR overbidding in distribution. Theoretically, we first compare RNBR overbidding, with the more common RNBNE overbidding assumption, which is found by a large literature. We show that RNBR overbidding implies RNBNE overbidding, but not vice versa. However, RNBR overbidding arises in many models that are consistent with RNBNE overbidding. In this sense, RNBR overbidding is not significantly more restrictive than RNBNE overbidding. Then we show that RNBR overbidding in distribution is weaker than the RNBR overbidding because it allows for underbidding as long as a sufficient fraction of bidders overbid. Empirically, we show that RNBR overbidding for all bidders does not hold in the experimental data from Dyer, Kagel, and Levin (1989) but RNBR overbidding in distribution does.

3.1 RNBR Overbidding Theory

An extensive literature finds that bidders tend to bid more than the RNBNE strategy. One way to assess the plausibility of the RNBR overbidding is to compare it with the RNBNE overbidding.

[8] We thank an anonymous referee for suggesting this discussion.

First, we derive the relationships that allows us to compare RNBNE overbidding and the RNBR overbidding under symmetry.

Proposition 1. *Suppose that all bidders adopt the same increasing strategy $s_n(\cdot)$ that satisfies rationality (Assumption 4).*

(1) If $s_n(\cdot) = \rho_n(\cdot)$, then $s_n(\cdot) = \rho_n(\cdot) = \sigma_n(\cdot)$.

(2) If $s_n(\cdot) = \sigma_n(\cdot)$, then $s_n(\cdot) = \sigma_n(\cdot) = \rho_n(\cdot)$.

(3) If $s_n(\cdot) \geq \rho_n(\cdot)$, then $s_n(\cdot) \geq \sigma_n(\cdot)$.

Notice that $\rho_n(\cdot) = \sigma_n(\cdot)$ does not imply $s_n(\cdot) = \rho_n(\cdot) = \sigma_n(\cdot)$.[9] The third part of Proposition 1 shows that overbidding compared to RNBR implies overbidding compared to RNBNE. However, $s_n(\cdot) \geq \sigma_n(\cdot)$ does not imply that $s_n(\cdot) \geq \rho_n(\cdot)$. See Example 1 for a counter-example. Intuitively, RNBNE overbidding only puts restrictions on the level of s_n. RNBR overbidding, however, puts restrictions on both the level and the derivative of s_n because the RNBR strategy depends on the latter through the bid density. Therefore, RNBR overbidding is stronger than RNBNE overbidding.

Example 1. Let F_n be the uniform distribution on $[0,1]$ and $n = 2$. Hence $\sigma_n(t) = \frac{t}{2}$. Now let $s_n(t) = 0.7t - 0.2t^2 \geq \sigma_n(t)$. One can check that $s_n(0.75) = 0.4125 < 0.4444 \approx \rho_n(0.75)$. Intuitively, s_n' becomes small for large t and therefore the best response is to reduce the bid shading because a slight increase in the bid increases the winning probability substantially. However, RNBNE overbidding is still satisfied because it does not put additional restrictions on s_n'.

Next, we show that RNBR overbidding is not much stronger than RNBNE overbidding in the sense that most commonly used auction models that are consistent with RNBNE overbidding are also consistent with RNBR overbidding. In the literature, there are two common ways to allow for deviations from RNBNE: deviating from the risk neutral utility function and deviating from rational expectations. In general, a bidder with valuation t in an n-bidder auction solves

$$\max_b U_n(t-b)\, C_n\left(F_n\left(s_n^{-1}(b)\right)^{n-1}\right). \tag{1}$$

Both U_n and C_n may deviate from the identity function. The former captures deviation from the risk neutral utility function and the latter allows for non-rational beliefs. Many common explanations for overbidding in the literature can be summarized by (1). For example, if bidders are risk averse, U_n is a concave function and $C_n(x) = x$. If bidders are ambiguity averse with Choquet expected utility, then $U_n(x) = x$ and C_n is a convex function with $C_n(0) = 0$. To characterize the equilibrium, one can take the first-order condition of (1) and impose symmetry, i.e., the solution to (1) is equal to $s_n(t)$. Define $\lambda_n(x) = U_n(x)/U_n'(x)$ and $H_n(t) = C_n\left(F_n^{n-1}(t)\right)/C_n'\left(F_n^{n-1}(t)\right)$. We can then obtain (2) in the following lemma, which generates RNBR overbidding with additional assumptions.

[9] For a counter-example let F be the uniform distribution on $[0,1]$ and $n = 2$. Now let $s_n(t) = ct$ for some constant $c \in (0.5, 1]$; then $\sigma_n(t) = \frac{t}{2} = \rho_n(t) < s_n(t)$.

Lemma 1. *Suppose there is no reserve price and bidders adopt a homogeneous strategy that satisfies the following: There exists a constant $\tilde{t}$ such that the bid strategy $s_n(t)$ satisfies: (1) if $t < \tilde{t}$, $s_n(t) = t$; (2) if $t \geq \tilde{t}$, s_n solves the following differential equation with initial condition $s_n\left(\tilde{t}\right) = \tilde{t}$,*

$$s'_n(t) = (n-1)\,\lambda_n\left(t - s_n(t)\right) H_n(t)\,\frac{f_n(t)}{F_n(t)} \tag{2}$$

where $H_n \geq 1$, $\lambda'_n(x) \geq 1$ and $\lambda_n(0) = 0$. Then bidders overbid compared to the RNBR.

Proof. If $t < \tilde{t}$, then

$$\rho_n^{-1}\left(s_n(t)\right) = t + \frac{F_n(t)}{(n-1)\,f_n(t)} \geq t,$$

which satisfies RNBR overbidding. If $t > \tilde{t}$, then $f_n(t)/s'_n(t) = g_n\left(s_n(t)\right)$ and $G_n\left(s_n(t)\right) = F_n(t)$. We can rewrite (2) as

$$\frac{1}{n-1}\frac{G_n\left(s_n(t)\right)}{g_n\left(s_n(t)\right)} = \lambda_n\left(t - s_n(t)\right) H_n(t) \geq t - s_n(t)\,,$$

where the inequality holds because $\lambda'_n \geq 1$ and $H_n \geq 1$. This implies that

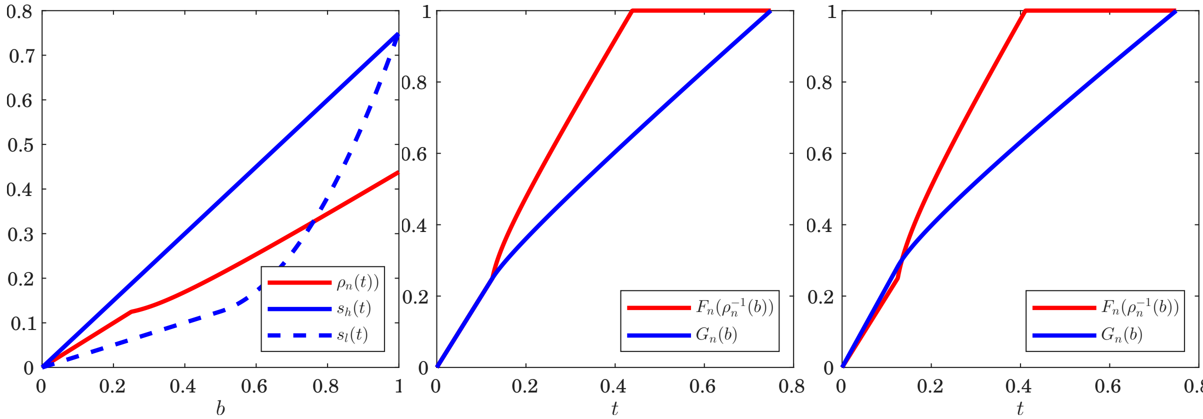
$$t \leq s_n(t) + \frac{1}{n-1}\frac{G_n\left(s_n(t)\right)}{g_n\left(s_n(t)\right)} = \rho_n^{-1}\left(s_n(t)\right),$$

or equivalently, $\rho_n(t) \leq s_n(t)$, i.e., bidders overbid compared to the RNBR. $\square$

The online appendix of Grundl and Zhu (2023a) shows that first-price auction models with risk aversion, loser regret, Choquet expected utility and loss aversion in money all satisfy the assumptions of Lemma 1.[10] In particular, s_n defined in Lemma 1 reduces to the RNBNE strategy if $\tilde{t} = 0$, $\lambda_n(x) = x$ and $H_n(t) = 1$. The model with MaxMin Expected utility also satisfies the assumptions if the lower contour of the prior set dominates the true valuation distribution in hazard rate. These are some of the most commonly used models to explain RNBNE overbidding and they all generate RNBR overbidding as well.

Proposition 2. *Under symmetry, first-price auction models with risk aversion, loser regret, Choquet expected utility or loss aversion in money satisfy RNBR overbidding.*

[10] For the loss aversion model, we need to have $\tilde{t} > 0$ and bidders gain only monetary benefits from winning the auction.



(a) Bid Functions and RNBR (b) RNBR Overbidding in Distribution (c) Violation of RNBR Overbidding in Distribution

Figure 1: RNBR Overbidding in Distribution

So far we have shown that the RNBR overbidding assumption is stronger than the RNBNE overbidding assumption but is still satisfied by many models that generate RNBNE overbidding under symmetry. We next show that the actual identifying assumption, i.e., RNBR overbidding in distribution (Assumption 3) does not require symmetric bidding strategies, and it does not even require every bidder to overbid compared to the RNBR. Instead, it can hold if a sufficiently high fraction of bidders overbid compared to the RNBR.

Example 2. Suppose that the auction has two bidders and their valuations are drawn from a uniform distribution between 0 and 1.[11] The two bidders randomly draw bidder types that determine their bidding strategies. With probability p, a bidder has a high type and uses $s_h(t) = 3t/4$, and with probability $1-p$, a bidder has a low type and uses $s_l(t) = t/4 + 2\max\{t-0.5, 0\}^2$. Then the bid distribution is

$$G_n(b) = \begin{cases} p\frac{4b}{3} + 4(1-p)\,b & \text{if } b < 1/8 \\ p\frac{4b}{3} + (1-p)\,\frac{7/2+\sqrt{32b-15/4}}{8} & \text{if } b > 1/8 \end{cases}$$

and the inverse of the RNBR strategy is

$$\rho_n^{-1}(b) = \begin{cases} 2b & \text{if } b < 1/8 \\ b + \frac{p\frac{4b}{3}+(1-p)\frac{7/2+\sqrt{32b-15/4}}{8}}{4p/3+2(1-p)/\sqrt{32b-15/4}} & \text{if } b > 1/8 \end{cases}.$$

Figure 1a plots s_h in solid blue, s_l in dashed blue, and ρ_n in red. Obviously, RNBR overbidding is violated because $s_l(t) < \rho_n(t)$ unless t is sufficiently large. But one can show that RNBR overbidding in distribution is still satisfied if and only if $p \geq 3/4$, i.e., at least 75% of bidders adopt s_h and overbid compared to the RNBR. Figure 1b shows $F_n\left(\rho_n^{-1}(b)\right)$ in red and $G_n(b)$ in blue when $p = 3/4$. In this case, $F_n\left(\rho_n^{-1}(b)\right) = G_n(b)$ if $b < 1/8$ and $F_n\left(\rho_n^{-1}(b)\right) > G_n(b)$ if $b > 1/8$.

[11] We thank an anonymous referee for suggesting a similar example.

Figure 1c shows the case with $p = 0.65$. As an insufficient share of bidders overbids, Assumption 3 now fails.

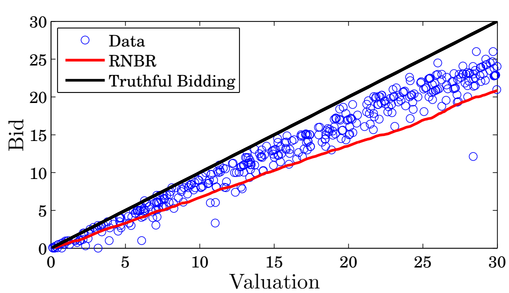
Lastly, we provide a formal result showing that RNBR overbidding in distribution is satisfied if a sufficiently large fraction of bidders overbid compared to RNBR. The idea is that we classify bidders into two groups. The two groups have the same valuation distribution. The first group overbids compared to the RNBR of their bid distribution and the second group may underbid compared to their bid distribution. Then the aggregate bid distribution is a mixture of the two bid distributions and it satisfies RNBR overbidding in distribution with the valuation distribution if the fraction of the first group is sufficiently high.

Proposition 3. *Suppose the valuation distribution is F_n and the bid distribution G_n is a mixture of $G_{n,1}$ and $G_{n,2}$ with weight on $G_{n,1}$ to be p. $G_{n,1}$, $G_{n,2}$ and F_n have densities with bounded derivatives that are strictly positive and continuous on their support. If $G_{n,1}$ can be rationalized by F_n and a bid function $s_{n,1}$ such that $s_{n,1}^{-1}(b) < \rho_{n,1}^{-1}(b)$ for all $b > 0$ in the support of $G_{n,1}$, and the support of $G_{n,2}$ is contained in that of $G_{n,1}$, then there exists $0 < \bar{p} < 1$ such that G_n satisfies RNBR overbidding in distribution if $p \geq \bar{p}$.*

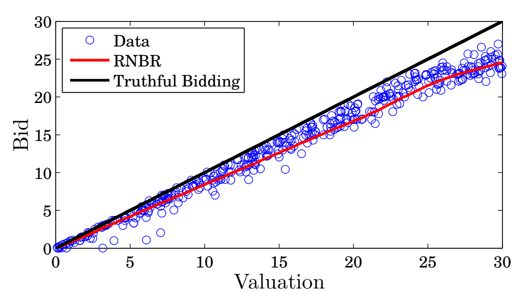
3.2 RNBR Overbidding in Experimental Data

The experimental literature has documented that bidders tend to overbid compared to the RNBNE strategy. However, little is known about whether the bidders overbid compared to RNBR. This is because the experimental literature is focused on comparing actual bidding behavior to the theoretical equilibrium prediction.

Proposition 1 shows that under symmetry RNBR overbidding implies RNBNE overbidding. In this sense RNBR overbidding is a stronger restriction than RNBNE overbidding. Therefore, the findings of RNBNE overbidding in the experimental literature do not justify the RNBR overbidding assumption.



(a) $n = 3$

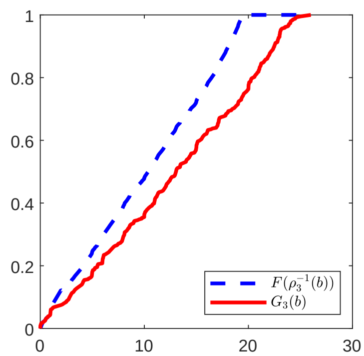


(b) $n = 6$

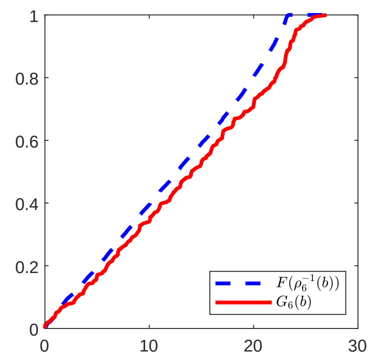
Figure 2: **Overbidding Compared to RNBR:** Experimental bid data from Dyer, Kagel, and Levin (1989).

Figure 2 shows experimental bid data from Dyer, Kagel, and Levin (1989). The data set contains contingent bids for $n = 3$ and $n = 6$ for the same underlying valuation from the same bidder. After discarding bids from training rounds there are 414 bids for $n = 3$ and 414 bids for $n = 6$. The valuations are drawn from a uniform distribution on $[0, 30]$.

The red line shows RNBR bidding and the black line shows "truthful" bidding. In three bidder auctions, 91.8 percent of the bidders overbid compared to RNBR. In six bidder auctions, 71.8 percent overbid. As n increases the space between RNBR bidding and truthful bidding shrinks so we observe more violations of RNBR overbidding. These findings suggest: (a) RNBR overbidding is predominant; (b) RNBR overbidding is not valid in the data because some bidders underbid.



(a) $n = 3$



(b) $n = 6$

Figure 3: **Overbidding in Distribution:** The actual bid distributions are shown in red and the bid distributions implied by RNBR are shown in blue.

Figure 3 shows that RNBR overbidding in distribution, seems to be satisfied in the bid data by Dyer, Kagel, and Levin (1989). While there is underbidding by some bidders especially for $n = 6$ the overall overbidding tendency seems to ensure that the bid distributions first-order stochastically dominate the bid distributions implied by RNBR bidding.

We next formally test RNBR overbidding in distribution. Notice that RNBR overbidding in distribution is equivalent to

$$\rho_n^{-1}\left(b_n\left(\alpha\right)\right) = b_n\left(\alpha\right) + \frac{\alpha}{\left(n-1\right) g_n\left(b_n\left(\alpha\right)\right)} \geq F_n^{-1}\left(\alpha\right),$$

where $b_n\left(\alpha\right) = G_n^{-1}\left(\alpha\right)$ is the quantile function of the bid distribution. Moreover, F_n is known from the experimental design and $F_n^{-1}\left(\alpha\right) = 30\alpha$. Let $A = \{\alpha_1, \alpha_2, \cdots, \alpha_J\}$ be a set of finite values in $[0, 1]$. We can formulate our hypothesis as

$$\begin{aligned} H_0 &: \rho_n^{-1}\left(b_n\left(\alpha\right)\right) \geq 30\alpha \text{ for all } \alpha \in A, \\ H_1 &: \rho_n^{-1}\left(b_n\left(\alpha\right)\right) < 30\alpha \text{ for some } \alpha \in A. \end{aligned}$$

We follow Marmer, Shneyerov, and Xu (2011) and define a test statistic as

$$\hat{T}_n = \left\lfloor \min_{\alpha \in A} \left\{ \frac{\hat{\rho}_n^{-1}\left(\hat{b}_n\left(\alpha\right)\right) - 30\alpha}{\hat{\Sigma}_n\left(\alpha\right)} \right\} \right\rfloor_{-}.$$

where $\lfloor a \rfloor_{-} = \min\{a, 0\}$,

$$\hat{\rho}_n^{-1}\left(\hat{b}_n(\alpha)\right) = \hat{b}_n(\alpha) + \frac{1}{n-1}\frac{\alpha}{\hat{g}_n\left(\hat{b}_n(\alpha)\right)},$$

$$\hat{b}_n(\alpha) = \min\left\{b : \hat{G}_n(b) \geq \alpha\right\},$$

and $\hat{g}_n$ is a kernel estimator for the bid density. We follow Marmer, Shneyerov, and Xu (2011) to calculate $\hat{\Sigma}_n(\alpha)$, which is an estimator for the standard deviation of $\hat{\rho}_n^{-1}\left(\hat{b}_n(\alpha)\right)$. To obtain the critical value, we use bootstrap with the statistic

$$\hat{T}_n^* = \left\lfloor \min_{\alpha \in A}\left\{\frac{\hat{\rho}_n^{*-1}\left(\hat{b}_n^*(\alpha)\right) - \hat{\rho}_n^{-1}\left(\hat{b}_n(\alpha)\right)}{\hat{\Sigma}_n^*(\alpha)}\right\}\right\rfloor_{-},$$

where $\hat{\rho}_n^{*-1}$, $\hat{b}_n^*$ and $\hat{\Sigma}_n^*(\alpha)$ are estimated from a bootstrap sample. Then the p-value can be calculated as the probability that T_n is less than or equal to $\hat{T}_n^*$. We reject the null hypothesis if the p-value is close to 0. To implement the test, we set A to be a set of 20 equally spaced values between 0.05 and 0.95 and use 5000 bootstrap draws.The p-value is above 0.99 for the 3-bidder sample and the 6-bidder sample.[12] Therefore, we cannot reject RNBR overbidding in our sample. This is not surprising because the point estimates satisfy the null hypothesis as shown in Figure 3.

4 Identification

Instead of working with the cumulative distribution functions, we focus on the quantile functions. Define $v_n(\alpha) = F_n^{-1}(\alpha)$ for $\alpha \in [0, 1]$. Because f_n has a support with a lower bound of 0, $v_n(0) = 0$. The inverse of an upper (lower) bound of $v_n(\cdot)$ is a lower (upper) bound of $F_n(\cdot)$. We first study the case in which there is no variation in n and then show how to exploit variation in n to tighten the bounds. In each case, we start with identifying the valuation distribution and then move to the seller's profit.

4.1 Bounds without Variation in n

Valuation Quantile Function First notice that rationality (Assumption 4) directly implies $v_n(\cdot) \geq b_n(\cdot)$. Therefore, $\underline{v}_n(\cdot) = b_n(\cdot)$ is a valid lower bound for $v_n(\cdot)$. This bound can be attained if the bidders have CRRA utility with a coefficient approaching 1, which implies it is a sharp bound.

[12] We use the Epanechenikov Kernel to estimate $\hat{g}_n$ and set the bandwidth to be $h_n = 1.06 \times std_n \times N_n^{-1/4}$ where $N_n = 414$ and std_n is the standard deviation of bids from n-bidder auctions. The variance is estimated by $\hat{\Sigma}_n(\alpha)^2 = 0.6\alpha^2/\hat{g}_n\left(\hat{b}_n(\alpha)\right)^3/(n-1)^2/N_n/h_n$. We have also tried the Guassian kernel and considered an alternative test following Gimenes and Guerre (2022), where $b_n(\alpha)$ and $b_n'(\alpha) = 1/g_n(b_n(\alpha))$ are estimated using quantile regression. Results are similar.

For the upper bound, notice that if the bidders are bidding the RNBR, then the GPV inverse implies the valuation that corresponds to $b_n(\alpha)$ for any $\alpha \in [0,1]$ is $\bar{v}_n(\alpha) = \rho_n^{-1}(b_n(\alpha))$. If bidders overbid compared to the RNBR strategy in distribution,

$$F_n(\bar{v}_n(\alpha)) \geq G_n(b_n(\alpha)) = \alpha.$$

Therefore, $\bar{v}_n(\alpha)$ is a valid upper bound for $v_n(\alpha)$. In fact, it is also a sharp bound, because we cannot rule out RNBR bidding.

Proposition 4. *If the bidders are rational (Assumption 4) and overbid compared to RNBR in distribution (Assumption 3), then $\underline{v}_n(\alpha) \leq v_n(\alpha) \leq \bar{v}_n(\alpha)$ and the bounds are sharp.*

It is worth pointing out that both assumptions and the arguments for the proposition do not depend on the symmetry in bidding strategies. Therefore, $\underline{v}_n(\alpha)$ and $\bar{v}_n(\alpha)$ remain valid if there are heterogeneous bidding strategies that are unobserved to the bidders. In this case, bidders are aware of heterogeneity in bidding strategies but do not know which bidding strategy a particular bidder is using. As a result, bidders use the marginal bid distribution to predict the winning probability.[13]

Before moving to the bounds on profit, we would like to compare $\bar{v}_n$ with the bound under RNBNE overbidding studied in Grundl and Zhu (2023a). To do so, we first define overbidding in distribution relative to the RNBNE.

Definition (RNBNE Overbidding in Distribution)**.** Bidders overbid compared to RNBNE in distribution: $F_n\left(\sigma_n^{-1}(\cdot)\right) \geq G_n(\cdot)$.

Notice that if a bid distribution can be rationalized by RNBR (RNBNE) overbidding in distribution, it can also be rationalized by RNBR (RNBNE) overbidding and a symmetric bidding strategy. Under symmetry, overbidding compared to RNBR is a stronger restriction than overbidding compared to RNBNE by Proposition 1. Therefore, assuming overbidding compared to RNBR in distribution should generate tighter bounds than assuming overbidding compared to RNBNE in distribution. This is formalized by the following corollary.

Corollary 1. *RNBR overbidding in distribution implies RNBNE overbidding in distribution. Consequently, the upper bound of the valuation quantile function under RNBR overbidding in distribution is weakly lower than the upper bound under RNBNE overbidding in distribution.*

Proof. Suppose that G_n can be rationalized by F_n under rationality and RNBR overbidding in distribution. Define $s_n(v) = G_n^{-1}[F_n(v)]$. Because $F_n\left(\rho_n^{-1}(b)\right) \geq G_n(b)$ implies $\rho_n(v) \leq G_n^{-1}[F_n(v)] = s_n(v)$, (F_n, s_n) rationalizes G_n under rationality and RNBR overbidding. By Proposition 1(3), $s_n(\cdot) \geq \sigma_n(\cdot)$, where $\sigma_n(\cdot)$ is the RNBNE bidding strategy under F_n. This implies that F_n can also rationalize G_n under RNBNE overbidding. This proves the first claim. It also implies that $\mathcal{F}^{BR} \subseteq \mathcal{F}^{BNE}$, where $\mathcal{F}^{BR}$ and $\mathcal{F}^{BNE}$ are the sets of value distributions that can rationalize G_n under RNBR overbidding in distribution and RNBNE overbidding in distribution, respectively. The

[13] See Section 7 for an extension with known bidder heterogeneity.

sharp lower bounds on the valuation distribution are the lower contours of $\mathcal{F}^{BR}$ and $\mathcal{F}^{BNE}$ under the corresponding overbidding restrictions. Because $\mathcal{F}^{BR} \subseteq \mathcal{F}^{BNE}$, the lower contour of $\mathcal{F}^{BR}$ is weakly higher than $\mathcal{F}^{BNE}$. In other words, RNBR overbidding leads to a higher lower bound on the value distribution or a lower upper bound on the value quantile function. $\square$

It is also worth noting that overbidding compared to the RNBNE is not sufficient for the GPV approach to be the upper bound on the valuation quantile function. To see this, notice that if $s_n(\cdot)$ satisfies RNBNE overbidding but $\rho_n(t) > s_n(t)$ for some t, which can happen by Example 1, then $t > \rho_n^{-1}(s_n(t))$. This implies that if $v_n(\alpha) = t$, $v_n(\alpha) > \rho_n^{-1}(s_n(v_n(\alpha))) = \bar{v}_n(\alpha)$, i.e., $\bar{v}_n(\alpha)$ is no longer a valid upper bound for $v_n(\alpha)$. In Example 1, one can check that $\bar{v}_n(0.75) \approx 0.7125 < 0.75 = v_n(0.75)$.

Corollary 2. *Under RNBNE overbidding in distribution, $\bar{v}_n(\alpha)$ is not necessarily an upper bound of $v_n(\alpha)$.*

Profits One downside of the partial identification approach is that it is difficult to conduct counterfactual analysis. This difficulty arises because the partial identification approach makes weak assumptions on how bidders behave under counterfactual scenarios. We show that under additional assumptions, we can still obtain bounds for the auctioneer's profit under a binding reserve price, which allows us to obtain the bounds for the optimal reserve price. Let $\pi_n(r)$ be the expected profit of the seller under a reserve price r. It can then be expressed as

$$\pi_n(r) = \int_r^\infty [s_n(t,r) - c]\, dF_n(t)^n + c, \tag{3}$$

where $s_n(t,r)$ is the bidding strategy under the reserve price r and c is the value of the auctioned good to the seller. We assume that c is known. Notice that if we observe auctions with rich variation in r, we can in principle directly estimate the profit as a function of r nonparametrically. Here we consider the case where such data is not available and we try to obtain the counterfactual profits for possible reserve prices.

We start with obtaining an upper bound for $\pi_n(r)$. We could proceed without introducing additional assumptions by using only the rationality assumption, but this leads to an upper bound that is not very informative. To obtain an informative upper bound, we therefore introduce the following additional restriction on $s_n(t,r)$ and let $s_n(t) = s_n(t,0)$ be the bid function without a reserve price.

Assumption 5. *Bidders' bidding strategies are homogeneous and satisfy the following conditions: Under a reserve price $r \geq 0$, (1) if $t < r$, the bidding strategy $s_n(t,r)$ satisfies $s_n(t,r) = 0$; (2) if $t \geq r$, s_n solves the following differential equation with initial condition $s_n(r,r) = r$,*

$$\frac{\partial s_n(t,r)}{\partial t} = (n-1)\,\lambda_n(t - s_n(t,r))\, H_n(t)\, \frac{f_n(t)}{F_n(t)} \tag{4}$$

where $H_n(\cdot) \geq 1$, $\lambda_n'(\cdot) \geq 1$ and $\lambda_n(0) = 0$.

This assumption extends the assumptions in Lemma 1, which imply overbidding compared to RNBR, to the case with a binding reserve price. Therefore it is a generalized overbidding model that nests various overbidding models in the literature. It is stronger than the assumption used in Grundl and Zhu (2023a), which only requires $H_n(\cdot) > 0$ and $\lambda_n'(\cdot) \geq 0$, but the latter only implies overbidding compared to the RNBNE not compared to the RNBR. The two possible sources of overbidding in this model are λ_n which alters bidder preferences from the risk neutral case and H_n, which allows for deviations from correct beliefs. With RNBNE bidding $\lambda_n(x) = x$ and $H_n(t) = 1$. Assumption 5 fails if bidders use heterogeneous bidding strategies, or if their preferences and/or beliefs differ from RNBNE in a way that leads to underbidding.

With a slight abuse of notation, we define $b_n(\alpha, r)$ to be the α-th quantile of the bid distribution under reserve price r. This means $b_n(\alpha) = b_n(\alpha, 0)$. Assumption 5 implies that for any α such that $v_n(\alpha) > r$,

$$\frac{\partial}{\partial \alpha} b_n(\alpha, r) = (n-1) \frac{\lambda_n[v_n(\alpha) - b_n(\alpha, r)]}{\alpha} \mathcal{H}_n(\alpha), \tag{5}$$

where $\mathcal{H}_n(\alpha) = H_n(v_n(\alpha)) \geq 1$. Because $\lambda_n'(\cdot) \geq 1$, this implies that $b_n(\alpha, r) \leq \bar{\sigma}_n(\alpha, r)$ (see Lemma 2 in the Online Appendix), where $\bar{\sigma}_n(\alpha, r)$ solves

$$\frac{\partial}{\partial \alpha} \bar{\sigma}_n(\alpha, r) = (n-1) \frac{\bar{v}_n(\alpha) - \bar{\sigma}_n(\alpha, r)}{\alpha},$$

with the initial condition $\bar{\sigma}_n\left(\bar{v}_n^{-1}(r), r\right) = r$ for all $\bar{v}_n(\alpha) > r$ and $\bar{\sigma}_n(\alpha, r) = 0$ if $\bar{v}_n(\alpha) < r$. Indeed, $\bar{\sigma}_n(\cdot, r)$ is the RNBNE bid quantile function under the reserve price r if the valuation quantile function is $\bar{v}_n$. It has a closed form expression

$$\bar{\sigma}_n(\alpha, r) = \begin{cases} 0 & \text{if } \bar{v}_n(\alpha) < r, \\ \frac{\bar{v}_n^{-1}(r)^{n-1} r}{\alpha^{n-1}} + \frac{n-2}{\alpha^{n-1}} \int_{r \leq \bar{v}_n(\alpha)}^{\alpha} x^{n-2} \bar{v}_n(x)\, dx & \text{if } \bar{v}_n(\alpha) \geq r, \end{cases} \tag{6}$$

where $\bar{v}_n^{-1}(r) = \inf\{x : \bar{v}_n(x) \geq r\}$. If the seller's valuation for the auctioned good is c, an upper bound of the profit as a function of r is

$$\bar{\pi}_n(r) \quad = \quad \int_{\bar{v}_n^{-1}(r)}^{1} \left[\bar{\sigma}_n(\alpha, r) - c\right] d\alpha^n + c. \tag{7}$$

It is worth noting that $\bar{\pi}_n$ is the profit function that one would obtain using the method developed in Guerre, Perrigne, and Vuong (2000). In other words, the common procedure used in the empirical auction literature does not only lead to an upper bound for the valuation quantile function but also to an upper bound for the profit function. Moreover, we will later show that this upper bound is sharp.

The lower bound for the profit function can be obtained by assuming that bidders bid truthfully, which leads to

$$\underline{\pi}_n(r) = \int_{b_n^{-1}(r)}^{1} [b_n(\alpha) - c]\, d\alpha^n + c. \tag{8}$$

This lower bound is sharp because it can be obtained under Assumption 5 if $H_n = 1$ and bidders have CRRA utility with the coefficient approaching 1. Therefore, the following proposition holds.

Proposition 5. *If bidders are rational (Assumption 4) and Assumption 5 holds, then* $\underline{\pi}_n(r) \leq \pi_n(r) \leq \bar{\pi}_n(r)$ *and the bounds are sharp.*

Proof. We only show that $\bar{\pi}_n(r)$ is a sharp upper bound. Applying Lemma 2 in the Online Appendix, we obtain $\bar{\sigma}_n(\alpha, r) \geq b_n(\alpha, r)$. Because $v_n(\cdot) \leq \bar{v}_n(\cdot)$

$$\pi_n(r) = \int_{v_n^{-1}(r)}^{1} [b_n(\alpha, r) - c]\, d\alpha^n + c \leq \int_{\bar{v}_n^{-1}(r)}^{1} [\bar{\sigma}_n(\alpha, r) - c]\, d\alpha^n + c = \bar{\pi}_n(r)$$

To see that this bound is sharp, notice that the observed bid quantile function $b_n(\cdot, 0)$ can be rationalized by $\bar{v}_n(\cdot)$ with RNBNE, in which case the upper bound is attained. □

Now we have shown that if bidders overbid compared to RNBR, the standard approach in the structural auction literature yields not only a sharp upper bound for the valuation quantile function but also a sharp upper bound for the profit function. Intuitively, there are many pairs of a valuation quantile function and a bidding model which are consistent with $b_n(\cdot)$ under $r = 0$. For $r = 0$ they all yield the same profit because they all generate the observed bid distribution. As r increases, the RNBNE bidding strategy increases most among all strategies that satisfy Assumption 5 because the RNBNE bidding strategy has the largest bid shading and therefore leaves the biggest room for more aggressive bidding. As a result, the profit function increases most under the RNBNE bidding strategy, implying that the RNBNE profit function is an upper bound. An implication of this result is that Guerre, Perrigne, and Vuong (2000) is a robust approach for choosing the reserve price in the sense that the resulting reserve price can be interpreted as a max-max reserve price, i.e. a reserve price maximizing the upper profit bound.

As pointed out before, the GPV valuation quantile function remains a valid upper bound under certain levels of heterogeneity in bidding strategies. We cannot formally prove that the GPV profit function has the same property. We have, however, run a range of numerical examples where there are a finite number of bidder types and bidders bid according to Assumption 5 but λ_n and H_n can vary across bidder types. In all considered examples, the profit function from Guerre, Perrigne, and Vuong (2000) remains as a valid upper profit bound, which suggests that the profit function may be robust to this type of bidding strategy heterogeneity. In Section 7, we propose alternative profit bounds that are proved to be valid in this case.

4.2 Tightening Bounds with Variation in n

In many applications, auctions with different numbers of bidders are observed. This variation in n can be used to tighten the bounds. To exploit this variation, we must restrict how the valuation distribution varies with n. To this end, we consider the following two assumptions:

Assumption 6. *$v_n(\cdot) = v(\cdot)$ for all $n \in \mathbb{N}$, where $\mathbb{N}$ is a finite set of integers.*

Assumption 7. *$v_n(\alpha)$ is weakly increasing in n for all $\alpha \in [0,1]$.*

Assumption 6 requires that the valuation distribution is independent of n. This assumption is sometimes called exogenous participation (EP) in the literature. It holds trivially if bidders are randomly assigned to auctions. However, even if the bidders decide which auctions to enter, it holds in many cases (Grundl and Zhu (2023b)). EP is used in many papers to either improve inference and/or to achieve identification.[14] Testing EP empirically in field data is usually difficult due to the presence of unobserved auction level heterogeneity.

Assumption 7 is usually referred to as stochastically increasing valuations (SIV). It allows for some positive selection and thereby relaxes Assumption 6. This assumption is also considered in Aradillas-López, Gandhi, and Quint (2011), Grundl and Zhu (2016) and Grundl and Zhu (2023a). Notice that we do not restrict how bidding behavior changes with n. This is important for robustness because different overbidding models have different predictions on how bidding changes with n.

It is straightforward to exploit variation in n to tighten the bounds of the valuation quantile function. We only need to obtain upper and lower bounds for each n and take the minimum of the upper bounds and the maximum of the lower bounds over the relevant set of auction sizes. Define $\bar{v}^{EP}(\alpha) = \min_{m \in \mathbb{N}} \bar{v}_m(\alpha)$, $\bar{v}_n^{SIV}(\alpha) = \min_{m \geq n} \bar{v}_m(\alpha)$, $\underline{v}^{EP}(\alpha) = \max_{m \in \mathbb{N}} \underline{v}_m(\alpha)$, $\underline{v}_n^{SIV}(\alpha) = \max_{m \leq n} \underline{v}_m(\alpha)$.

Proposition 6. *Suppose bidders are rational (Assumption 4) and overbid compared to RNBR in distribution (Assumption 3).*

(1) If Assumption 6 holds, then $\underline{v}^{EP}(\alpha) \leq v_n(\alpha) \leq \bar{v}^{EP}(\alpha)$.

(2) If Assumption 7 holds, then $\underline{v}_n^{SIV}(\alpha) \leq v_n(\alpha) \leq \bar{v}_n^{SIV}(\alpha)$.

The upper profit bound can be obtained by a slight change to the result without variation in n. Define

$$\begin{aligned} \bar{\pi}_n^{EP}(r) &= \int_{\bar{v}^{EP}(\alpha) \geq r}^{1} \left[\min\left\{\bar{v}^{EP}(\alpha), \bar{\sigma}_n(\alpha, r)\right\} - c\right] d\alpha^n + c, \\ \bar{\pi}_n^{SIV}(r) &= \int_{\bar{v}_n^{SIV}(\alpha) \geq r}^{1} \left[\min\left\{\bar{v}_n^{SIV}(\alpha), \bar{\sigma}_n(\alpha, r)\right\} - c\right] d\alpha^n + c. \end{aligned}$$

[14] An incomplete list includes Haile and Tamer (2003), Haile, Hong, and Shum (2003), Guerre, Perrigne, and Vuong (2009), Aradillas-López, Gandhi, and Quint (2011), Aryal, Grundl, Kim, and Zhu (2018) and Grundl and Zhu (2023a).

Several comments are in order. First, we use the RNBNE bid quantile under $\bar{v}_n$, $\bar{\sigma}_n(\alpha, r)$, instead of the RNBNE bid quantile under $\bar{v}^{EP}$ or $\bar{v}_n^{SIV}$ to form the upper bound. This is because the latter may not be a valid upper bound. Second, the counterfactual bidding strategy should lie below the upper bound for the valuation quantile function by rationality. Because $\bar{\sigma}_n(\alpha, r)$ is constructed under $\bar{v}_n$, it is not guaranteed to lie below $\bar{v}^{EP}$ or $\bar{v}_n^{SIV}$. Therefore, we take the minimum of $\bar{\sigma}_n(\alpha, r)$ and $\bar{v}^{EP}$ or $\bar{v}_n^{SIV}$. Lastly, using variation in n improves the upper bound through adding the minimum in the integrant and shrinking the region of integration because both $\bar{v}^{EP}$ and $\bar{v}_n^{SIV}$ are in general lower than $\bar{v}_n$.

To obtain the lower profit bound, define $\underline{\sigma}_n^{EP}(\alpha, r)$ and $\underline{\sigma}_n^{SIV}(\alpha, r)$ to be the RNBNE bid quantile function under $\underline{v}^{EP}(\cdot)$ and $\underline{v}_n^{SIV}(\cdot)$. Then the lower bounds are

$$\begin{aligned}
\underline{\pi}_n^{EP}(r) &= \int_{\underline{v}^{EP}(\alpha) \geq r}^{1} \left[\max\left\{b_n(\alpha), \underline{\sigma}_n^{EP}(\alpha, r)\right\} - c\right] d\alpha^n + c \\
\underline{\pi}_n^{SIV}(r) &= \int_{\underline{v}_n^{SIV}(\alpha) \geq r}^{1} \left[\max\left\{b_n(\alpha), \underline{\sigma}_n^{SIV}(\alpha, r)\right\} - c\right] d\alpha^n + c.
\end{aligned}$$

Because bidders overbid compared to the RNBNE bidding strategy under Assumption 5, $b_n(\alpha, r)$ is no less than $\underline{\sigma}_n^{EP}(\alpha, r)$ or $\underline{\sigma}_n^{SIV}(\alpha, r)$. Moreover, an implication of Assumption 5 is that a higher reserve price leads to higher bids from bidders whose valuation is higher than the reserve price. Therefore, if $r > 0$, $b_n(\alpha)$ is no larger than $b_n(\alpha, r)$ for all sufficiently large α.

Proposition 7. *If bidders are rational (Assumption 4) and Assumption 5 holds.*

(1) If Assumption 6 holds, then $\underline{\pi}_n^{EP}(r) \leq \pi_n(r) \leq \bar{\pi}_n^{EP}(r)$.

(2) If Assumption 7 holds, then $\underline{\pi}_n^{SIV}(r) \leq \pi_n(r) \leq \bar{\pi}_n^{SIV}(r)$.

Notice that all the bounds in Proposition 7 are not sharp but they are very easy to compute. To obtain the sharp bounds, one needs to solve a high dimensional nonlinear optimization problem as in Grundl and Zhu (2023a), which is computationally difficult and the improvement is limited. Therefore, we do not pursue the sharp bounds here.

The identification results we established so far are valid if bidders have a general tendency to overbid compared to the RNBR. They may not be valid if the majority of bidders bid lower than the RNBR. A practical question is how to detect overbidding compared to the RNBR. The following result can provide some answer to this question.

Proposition 8. *If Assumption 6 holds and Assumption 5 holds with λ_n and H_n constant across n, then $\rho_n^{-1}(b_n(\alpha))$ is weakly decreasing in n for all $\alpha \in [0, 1]$.*

Proof. Because v_n and H_n are independent of n, so is $\mathcal{H}_n$. Therefore, we suppress the subscripts of v_n, λ_n and $\mathcal{H}_n$. Notice that for any $n \in \mathbb{N}$,

$$\rho_n^{-1}(b_n(\alpha)) - b_n(\alpha) = \frac{\alpha b_n'(\alpha)}{n-1} = \lambda(v(\alpha) - b_n(\alpha))\mathcal{H}(\alpha), \tag{9}$$

where the second equality follows from Assumption 5. Take the difference of (9) evaluated at $n = n_1$ and n_2, where $n_1 < n_2$. Then we obtain

$$\rho_{n_1}^{-1}(b_{n_1}(\alpha)) - b_{n_1}(\alpha) - \rho_{n_2}^{-1}(b_{n_2}(\alpha)) + b_{n_2}(\alpha) = \lambda(v(\alpha) - b_{n_1}(\alpha))\mathcal{H}(\alpha) - \lambda(v(\alpha) - b_{n_2}(\alpha))\mathcal{H}(\alpha).$$

One can show that $b_{n_2}(\alpha) > b_{n_1}(\alpha)$ for all $\alpha \in (0,1]$. Because $\lambda' \geq 1$ and $\mathcal{H} \geq 1$,

$$\lambda(v(\alpha) - b_{n_1}(\alpha))\mathcal{H}(\alpha) - \lambda(v(\alpha) - b_{n_2}(\alpha))\mathcal{H}(\alpha) \geq b_{n_2}(\alpha) - b_{n_1}(\alpha).$$

This implies that

$$\rho_{n_1}^{-1}(b_{n_1}(\alpha)) - b_{n_1}(\alpha) - \rho_{n_2}^{-1}(b_{n_2}(\alpha)) + b_{n_2}(\alpha) \geq b_{n_2}(\alpha) - b_{n_1}(\alpha),$$

or equivalently

$$\rho_{n_1}^{-1}(b_{n_1}(\alpha)) - \rho_{n_2}^{-1}(b_{n_2}(\alpha)) \geq 0,$$

for all $\alpha \in [0,1]$. This proves the proposition. □

This proposition implies that we can apply the GPV estimator to the data and if the resulting valuation quantile function is decreasing in n for some α, one may suspect that there is overbidding. If the bidders would use RNBNE bidding the recovered distributions would be indistinguishable.[15] If Assumption 7 holds instead of Assumption 6 but λ_n and H_n remain independent of n, this insight is further strengthened: if the valuations are increasing in n, it would be even less likely that the valuation quantile functions recovered using GPV are decreasing in n.

5 Estimation

Suppose we observe N_n bids, $\{y_{i,n}\}_{i=1}^{N_n}$, from n-bidder auctions for each $n \in \mathbb{N}$. Let $N = \sum_{n\in\mathbb{N}} N_n$ be the total number of observations and assume that $\lim_{N\to\infty} N_n/N > 0$ for every $n \in \mathbb{N}$. The goal is to estimate the bounds for the valuation quantile function and the profit function.

Let $\hat{b}_n(\cdot)$ be an estimator for the bid quantile function $b_n(\cdot)$, $\hat{G}_n(\cdot)$ be an estimator for the bid distribution function $G_n(\cdot)$, and $\hat{g}_n(\cdot)$ be an estimator for the density function $g_n(\cdot)$. We can estimate the GPV inverse bidding function by

$$\widehat{\rho_n^{-1}}(b) = b + \frac{\hat{G}_n(b)}{(n-1)\hat{g}_n(b)}. \tag{10}$$

The estimator of the upper bound for the valuation quantile function in an n-bidder auction is

[15] One can show that Assumption 5 with $\lambda_n' \leq 1$ and $H_n \leq 1$ leads to underbidding compared to the RNBR strategy and then a similar argument as the proof of Proposition 8 shows that $\rho_n^{-1}(b_n(\alpha))$ is weakly increasing in n for all $\alpha \in [0,1]$ if v_n, λ_n and H_n are independent of n. Therefore, if bidders underbid, we expect the GPV valuation quantile to be increasing with n instead of decreasing with n.

defined as

$$\hat{\bar{v}}_n(\alpha) = \inf\left\{x : \hat{\underline{F}}_n(x) \geq \alpha\right\} \tag{11}$$

where $\hat{\underline{F}}_n(x) = \sum_{i=1}^{N_n} \mathbf{1}\left(\widehat{\rho_n^{-1}}(y_{i,n}) \leq x\right)/N_n$. Notice that we estimate the valuation quantile function by inverting an estimator of the valuation distribution. This ensures that $\hat{\bar{v}}_n(\alpha)$ is increasing in α. Alternatively, one may estimate $\bar{v}_n(\alpha)$ by $\widehat{\rho_n^{-1}}\left(\hat{b}_n(\alpha)\right)$, but the resulting estimator may not be increasing. The lower bound of the valuation quantile function can be estimated by $\hat{\underline{v}}_n(\alpha) = \hat{b}_n(\alpha)$.

To estimate the bounds for the profit function, we just plug in the estimators of the bounds for the valuation quantile function into (7) and (8):

$$\hat{\bar{\pi}}_n(r) = \int_{\hat{\bar{v}}_n(\alpha) \geq r}^{1} \left[\hat{\bar{\sigma}}_n(\alpha, r) - c\right] d\alpha^n + c + \Delta_n,$$

$$\hat{\underline{\pi}}_n(r) = \int_{\hat{\underline{v}}_n(\alpha) \geq r}^{1} \left[\hat{b}_n(\alpha) - c\right] d\alpha^n + c,$$

where $\hat{\bar{\sigma}}_n(\alpha, r)$ is obtained by replacing $\bar{v}_n$ in (6) with $\hat{\bar{v}}_n$ and Δ_n solves $\hat{\bar{\pi}}_n(0) = \hat{\underline{\pi}}_n(0)$. The term Δ_n is introduced to ensure that the lower and the upper profit bounds coincide and are equal to the profit implied by the observed bid distribution if there is no reserve price. This holds as the sample size approaches to infinity because $\bar{\sigma}_n(\alpha, 0) = b_n(\alpha)$ implies $\bar{\pi}_n(0) = \underline{\pi}_n(0)$. But in finite samples, $\hat{\bar{\sigma}}_n(\alpha, 0)$ is in general different from $\hat{b}_n(\alpha)$ because of estimation errors. Consequently, the upper profit bound estimate can be lower than that of the lower profit bound without introducing Δ_n.

If Assumption 6 is imposed, we estimate the upper and lower bounds on the valuation quantile function by

$$\hat{\bar{v}}^{EP}(\alpha) = \min_{m \in \mathbb{N}} \hat{\bar{v}}_m(\alpha),$$

$$\hat{\underline{v}}^{EP}(\alpha) = \max_{m \in \mathbb{N}} \hat{\underline{v}}_m(\alpha).$$

The upper and lower bounds on the profit function can be estimated by

$$\hat{\bar{\pi}}_n^{EP}(r) = \int_{\hat{\bar{v}}^{EP}(\alpha) \geq r}^{1} \left[\min\left\{\hat{\bar{v}}^{EP}(\alpha), \hat{\bar{\sigma}}_n(\alpha, r)\right\} - c\right] d\alpha^n + c + \Delta_n,$$

$$\hat{\underline{\pi}}_n^{EP}(r) = \int_{\hat{\underline{v}}^{EP}(\alpha) \geq r}^{1} \left[\max\left\{\hat{b}_n(\alpha), \hat{\underline{\sigma}}_n^{EP}(\alpha, r)\right\} - c\right] d\alpha^n + c,$$

where $\hat{\underline{\sigma}}_n^{EP}(\alpha, r)$ is the RNBNE bid quantile under $\hat{\underline{v}}^{EP}(\alpha)$, which is calculated by replacing $\bar{v}_n(\alpha)$ by $\hat{\underline{v}}^{EP}(\alpha)$ in (6). Again, Δ_n solves $\hat{\bar{\pi}}_n^{EP}(0) = \hat{\underline{\pi}}_n^{EP}(0)$. If Assumption 7 holds, we estimate the bounds in an analogous way. Denote the resulting estimators for the upper and lower bounds for the valuation quantile function as $\hat{\bar{v}}_n^{SIV}(\alpha)$ and $\hat{\underline{v}}_n^{SIV}(\alpha)$, and the estimators for the profit function bounds as $\hat{\bar{\pi}}_n^{SIV}(r)$ and $\hat{\underline{\pi}}_n^{SIV}(r)$.

To derive the asymptotic properties of the estimators, we introduce the following high-level assumption, where $\underline{b}_n$ and $\bar{b}_n$ are the lower and upper bounds of the support of $g_n(\cdot)$, respectively.

Assumption 8. *For all $n \in \mathbb{N}$, the estimators $\hat{b}_n(\cdot)$, $\hat{G}_n(\cdot)$and $\hat{g}_n(\cdot)$ satisfy that as $N \to \infty$,*

$$\sup_{\alpha\in[0,1]} \left|\hat{b}_n(\alpha) - b_n(\alpha)\right| \to^p 0,$$
$$\sup_{b\in\left[\underline{b}_n,\bar{b}_n\right]} \left|\hat{G}_n(b) - G_n(b)\right| \to^p 0,$$

and *for every sufficiently small $\varepsilon > 0$,*

$$\sup_{b\in B_n(\varepsilon)} \left|\hat{g}_n(b) - g_n(b)\right| \to^p 0,$$

where $B_n(\varepsilon) = \left[\underline{b}_n, \bar{b}_n\right]$ if $g_n(\cdot)$ is continuous on its support and

$$B_n(\varepsilon) = \left[\underline{b}_n, b_{n,1} - \varepsilon\right] \cup \left[b_{n,L} + \varepsilon, \bar{b}_n\right] \cup \left\{\cup_{l=2}^{L-1}\left[b_{n,l-1} + \varepsilon, b_{n,l} - \varepsilon\right]\right\}$$

if $g_n(\cdot)$ has L jumps at $b_{n,1} < b_{n,2} < \cdots < b_{n,L}$ in $\left(\underline{b}_n, \bar{b}_n\right)$.

Assumption 8 allows the bid density to have a finite number of jumps, which could happen if bidders adopt heterogeneous bidding strategies. For example, suppose bidders are risk averse with constant relative risk aversion (CRRA) utility. The CRRA coefficient is a random variable that can take a finite number of values. If bidders best respond to the marginal bid distribution, then the resulting bid density has jumps.

Define $A_n(\varepsilon) = [0, 1]$ if g_n is continuous on its support. If g_n has L jumps at $b_{n,1} < b_{n,2} < \cdots < b_{n,L}$, define $\alpha_{n,l} = G_n(b_{n,l})$ for $l = 1, 2, \cdots, L$ and

$$A_n(\varepsilon) = [0, \alpha_{n,1} - \varepsilon] \cup [\alpha_{n,L} + \varepsilon, 1] \cup \left\{\cup_{l=2}^{L-1}\left[\alpha_{n,l-1} + \varepsilon, \alpha_{n,l} - \varepsilon\right]\right\}.$$

Proposition 9. *If Assumption 8 holds, then for every $\varepsilon > 0$*

$$\sup_{\alpha\in[0,1]} \left|\hat{\underline{v}}_n(\alpha) - \underline{v}_n(\alpha)\right| \to^p 0 \text{ and } \sup_{\alpha\in A_n(\varepsilon)} \left|\hat{\bar{v}}_n(\alpha) - \bar{v}_n(\alpha)\right| \to^p 0,$$
$$\sup_{\alpha\in[0,1]} \left|\hat{\underline{v}}^{EP}(\alpha) - \underline{v}^{EP}(\alpha)\right| \to^p 0 \text{ and } \sup_{\alpha\in\cap_{m\in\mathbb{N}}A_m(\varepsilon)} \left|\hat{\bar{v}}^{EP}(\alpha) - \bar{v}^{EP}(\alpha)\right| \to^p 0,$$
$$\sup_{\alpha\in[0,1]} \left|\hat{\underline{v}}_n^{SIV}(\alpha) - \underline{v}_n^{SIV}(\alpha)\right| \to^p 0 \text{ and } \sup_{\alpha\in\cap_{m\geq n}A_m(\varepsilon)} \left|\hat{\bar{v}}_n^{SIV}(\alpha) - \bar{v}_n^{SIV}(\alpha)\right| \to^p 0.$$

Moreover, for all $r \geq 0$ and $n \in \mathbb{N}$

$$\left|\hat{\underline{\pi}}_n(r) - \underline{\pi}_n(r)\right| \to^p 0 \text{ and } \left|\hat{\bar{\pi}}_n(r) - \bar{\pi}_n(r)\right| \to^p 0,$$
$$\left|\hat{\underline{\pi}}_n^{EP}(r) - \underline{\pi}_n^{EP}(r)\right| \to^p 0 \text{ and } \left|\hat{\bar{\pi}}_n^{EP}(r) - \bar{\pi}_n^{EP}(r)\right| \to^p 0,$$
$$\left|\hat{\underline{\pi}}_n^{SIV}(r) - \underline{\pi}_n^{SIV}(r)\right| \to^p 0 \text{ and } \left|\hat{\bar{\pi}}_n^{SIV}(r) - \bar{\pi}_n^{SIV}(r)\right| \to^p 0.$$

There are several estimators in the literature that satisfy the conditions of Proposition 9. One commonly used method that has proved to work well is to define $\hat{G}_n(b) = \sum_{i=1}^{N_n} \mathbf{1}(y_{i,n} \leq b)/N_n$, $\hat{b}_n$ to be its inverse

$$\hat{b}_n(\alpha) = \inf\left\{b : \hat{G}_n(b) \geq \alpha\right\}$$

and define $\hat{g}_n$ to be the boundary corrected kernel estimator proposed by Hickman and Hubbard (2015).[16] Then Assumption 8 holds under mild conditions. In particular, even if g_n has jumps, $\hat{g}_n$ is uniformly consistent if we exclude neighborhoods of the jump points. This approach does not require optimization and therefore is easy to compute. We will focus on this approach in this paper.

There are, however, other methods that also work well in this case, which include the method considered in Zincenko (2024), the quantile regression based approach proposed by Gimenes and Guerre (2022), and the functional approach proposed by Enache, Florens, and Sbai (2023). Zincenko (2024) proposes a new boundary correction method for the inverse bidding function and obtains estimators for the valuation distribution and the profit function under RNBNE. Gimenes and Guerre (2022) estimate $b_n(\alpha)$ and $1/g_n(b_n(\alpha))$ simultaneously using an integrated quantile regression and can allow for covariates to capture auction heterogeneity. This approach requires solving a convex minimization problem and is computationally more challenging than the first approach we described, but is still manageable. To ensure monotonicity of the estimator for $\bar{v}_n$, one may use the rearrangement approach proposed by Chernozhukov, Fernandez-Val, and Galichon (2010). Enache, Florens, and Sbai (2023) propose one-step estimators of the valuation distribution or quantile function under RNBNE and hence can be used to estimate the upper bound of the valuation quantile function under RNBR overbidding. Some of their estimators involve solving a functional equation by iteration. We have tried the approach by Gimenes and Guerre (2022) and the quantile kernel approach by Enache, Florens, and Sbai (2023) and found similar results to using the Hickman and Hubbard (2015) boundary correction. We do not further discuss the finite-sample properties of the estimators because they have been well-studied in the literature.

6 Applications

We illustrate our approach in two applications. First, in Section 6.1 we use the experimental data from Dyer, Kagel, and Levin (1989) and compare the valuation bounds based on the assumption of RNBR overbidding to the bounds generated by the assumption of RNBNE overbidding (Grundl and Zhu, 2023a). The main advantage of using experimental data is that we know the actual valuation distribution and can compare it to the estimated bounds. Second, in Section 6.2, we apply the approach to field data from timber auctions. There we focus on how to tighten the bounds using variation in the number of bidders. We also estimate the bounds for the seller's profit and explain how the possibility of overbidding can rationalize why seller's may be cautious about setting high reserve prices.

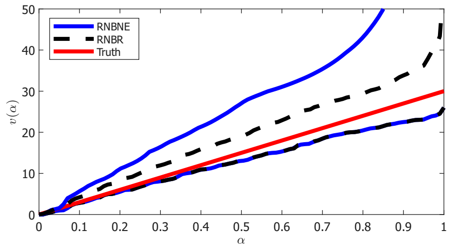
[16] One can ignore the potential discontinuities of g_n and still obtain consistent estimators. Or one can first identify the discontinuities in the density function, and then applies kernel density estimation separately on each subinterval where the density remains continuous.

In both applications we follow the implementation of the boundary corrected estimator proposed by Hickman and Hubbard (2015). We use the Triweight kernel and the rule-of-thumb bandwidth (Silverman, 1986) with the constant for the Triweight kernel (Härdle, 1991) to estimate the bid densities.[17]

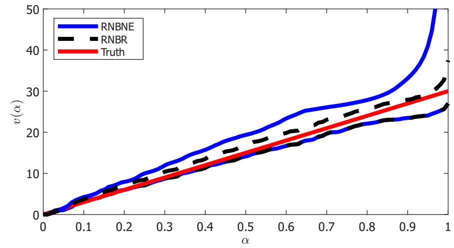
6.1 Experimental Data

Figure 4 shows the estimated bounds for the valuation quantile function for $n = 3$ on the left and $n = 6$ on the right using the experimental data from Dyer, Kagel, and Levin (1989). Panels (a) and (b) do not use variation in n. Panels (c) and (d) use variation in n under the SIV assumption (Assumption 7), and Panel (e) under the EP assumption (Assumption 6). The true valuation distribution is shown in red, the RNBNE bounds in blue, and the RNBR bounds in black.

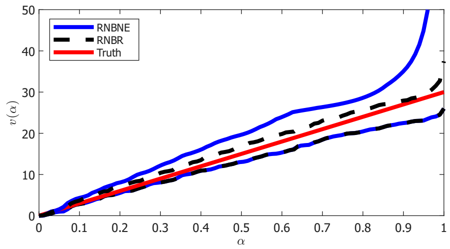
[17] Besides the kernel and the main bandwidth the method requires two more parameter choices, A and h_1, that Hickman and Hubbard (2015) describe as having little effect within certain bounds. First the constant A has no effect if $A > 1/3$ and is set to $A = 0.55$. Second, the bandwidth h_1 is set to $h_1 = hN_n^{-\frac{1}{20}}$, where h is the main bandwidth.



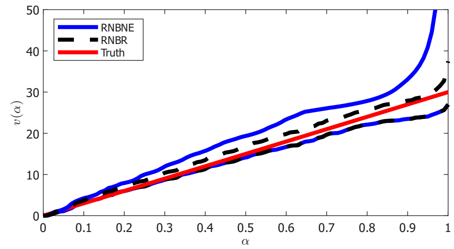
(a) $n = 3$



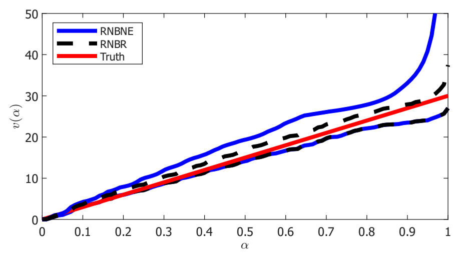
(b) $n = 6$



(c) SIV $n = 3$



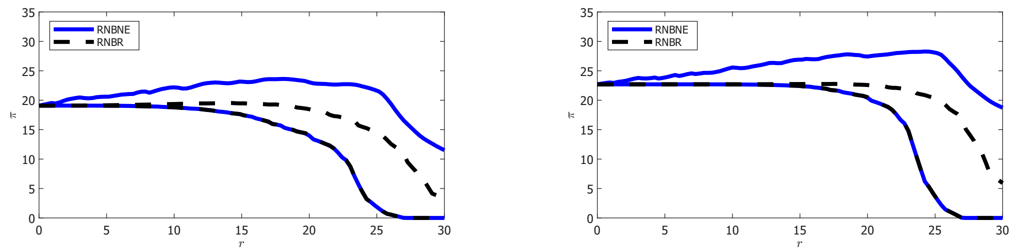
(d) SIV $n = 6$



(e) EP $n = 3$ and $n = 6$

Figure 4: **Bounding the Valuation Distribution (Laboratory Data):** Bounds on the valuation quantile function for $n = 3$ and $n = 6$. Panels (a) and (b) do not use variation in n. Panels (c) and (d) use variation in n under the SIV assumption, and Panel (e) under the EP assumption.

In all cases the RNBR bounds contain the true valuation quantile function. Moreover, consistent with Corollary 1, they are tighter than the RNBNE bounds. As we can see, at least in this data set, the RNBR bounds are substantially tighter, especially for higher quantiles. Indeed, the upper bound under RNBNE overbidding diverges to infinity as α converges to 1, as shown by Grundl and Zhu (2023a), while the upper bound under RNBR remains bounded.



(a) $n = 3$ (b) $n = 6$

Figure 5: **Bounding the Seller's Profit (Laboratory Data):** Bounds on the seller's profit for $n = 3$ and $n = 6$ under the EP assumption.

Figure 5 shows the estimates for the seller's profit under the EP assumption (Assumption 6) and with $c = 0$. Mirroring the estimates for the valuation distribution the RNBR bounds are considerably tighter than the RNBNE bounds.

6.2 Timber Data

As an illustrative application to field data we consider US Forest Service (USFS) auctions, which have been studied extensively in the structural auction literature.[18] We only use sales that were set aside for small businesses with fewer than 500 employees, which allows us to treat the bidders as symmetric. We focus on sales with 2-5 bidders, because variation in the number of bidders tends to be most informative as we move from two or three bidders to four or five bidders, but becomes less informative as competition increases further. Geographically, we focus on auctions in California. This leaves us with 1075 bids in total: 216 from two bidder auctions, 291 from 3 bidder auctions, 308 from four bidder auction, and 260 bids from five bidder auctions. Bids are measured in dollars per one thousand board feet of timber.

We homogenize the bids by regressing log bids on a number of covariates and a set of indicator variables for the number of bidders.[19] The most important timber tract characteristic is the appraisal value calculated by the USFS, which is designed to capture all relevant information about the tract. In addition, we also include the total timber volume on the tract, the timber density (volume per acre), the timber species concentration (HHI), a measure of backlog (volume of timber sold in the previous six months), and the species composition (shares of pine, fir, and other species). We construct the homogenized log bids as the sum of the residuals and the estimated intercepts for each number of bidders.

Before we apply our approach we first ask whether any pattern in the data suggests that there might be overbidding. In Proposition 8 we showed theoretically that if the estimated valuation

[18] See Baldwin, Marshall, and Richard (1997) for a description of the auction procedure and the timber industry.

[19] We use homogenized bids to simplify the analysis and to focus on illustrating the bounds approach. We could also use the estimator developed by Gimenes and Guerre (2022) to allow for more flexible dependence between the valuations and the observables.

quantiles under the RNBNE assumption are shifted to the left as the number of bidders increases this could be indicative of overbidding. Intuitively this is the case because overbidding tends to matter less in auctions with more bidders where RNBNE bidding is already close to the valuations.[20] Figure 6 shows the estimated valuation quantile functions under the assumption of RNBNE bidding for each n. While the estimates in Figure 6 do not match the predicted pattern in Proposition 8 perfectly because the estimates for $n = 3$, $n = 4$ and $n = 5$ have no clear order, it is striking that the estimate for $n = 2$ is clearly the highest. This pattern is not conclusive of overbidding but it suggests that applying an approach that is robust to overbidding might be prudent.[21]

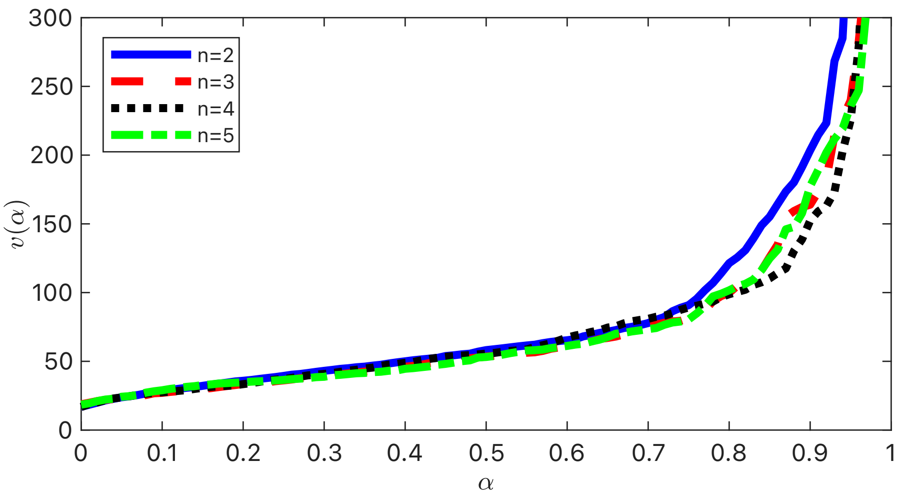


Figure 6: **Estimated Valuation Distributions for Different n (Timber Data):** This figure shows the estimated valuation distributions under the RNBNE assumption for different numbers of bidders.

Next we show estimates of the bounds under the assumption of RNBR overbidding in distribution. Figure 7 shows the bounds for the valuation distribution and Figure 8 shows the bounds for the profit assuming that the seller's valuation is $c = 0$. Each figure shows bounds without

[20] To see this, start with the baseline case where the actual valuation distributions are identical for each n and the bidders are playing the RNBNE strategy. Now suppose the bidders are risk averse. Risk aversion shifts the bids upwards towards the valuations but less so for auctions with more bidders where even the risk neutral bids are already close to the valuations. Therefore, risk aversion pushes the bid distributions for different n closer together. If we now invert these bid distributions under the assumption of RNBNE bidding the recovered valuation distributions would be highest for $n = 2$, followed by $n = 3$, $n = 4$ and $n = 5$.

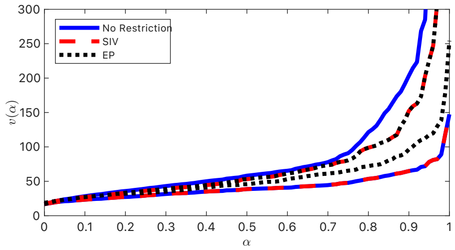
[21] This pattern is also consistent with the fact that several papers on timber auctions, but not all, found evidence of risk aversion. Lu and Perrigne (2008) use variation in the auction format to estimate bidder risk aversion nonparametrically and find evidence of risk aversion. Campo, Guerre, Perrigne, and Vuong (2011) use a different identification strategy and also find evidence of risk aversion. Grundl and Zhu (2019) find however that the data is consistent with risk neutrality once unobserved auction level heterogeneity is taken into account. Gimenes and Guerre (2022) find the the estimator of the coefficient of relative risk aversion has a high variance and risk neutrality cannot be rejected. Aryal, Grundl, Kim, and Zhu (2018) find evidence of ambiguity aversion in USFS timber auctions, which in this case also led to overbidding.

using variation in the number of bidders as a solid blue line, bounds that use the SIV assumption (Assumption 7) as a red dashed line, and the bounds using the EP assumption (Assumption 6) as a black dotted line. Using variation in the number of bidders leads to substantial tightening of the bounds especially for auctions with two bidders.

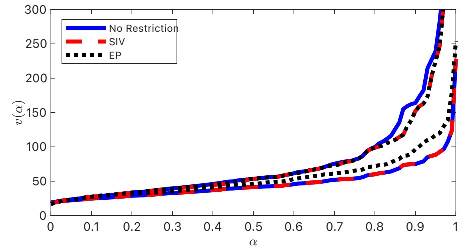
The gap between the profit bounds widens as the reserve price increases. For the lowest reserve price ($r = 16$), which is not binding for any bidder, the profit simply equals the observed profit in the data. Therefore both bounds coincide at that point. For higher reserve prices however the extent of overbidding becomes important. The lower profit bound assumes that the bidders did not shade their bids – the largest extent of overbidding – so the lower bound drops quickly as r increases, whereas the upper bound remains relatively stable. For the seller the expected profit for higher values of r is therefore more uncertain.[22] A cautious seller may therefore prefer to set a low reserve price or no reserve price at all even though the maximum of the upper bound is achieved at substantially higher reserve prices. Setting a higher reserve price could result in a modest profit increase but also in a substantial decrease of profits. Uncertainty about the degree of overbidding can therefore help to explain why sellers often set no reserve prices or reserve prices that are substantially lower than the RNBNE model would predict.[23]

[22] See Aryal and Kim (2013) for how to choose a reserve price in a partially identified auction model.

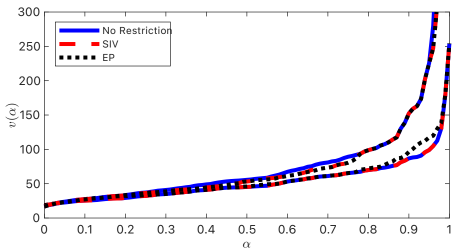
[23] See Hu, Matthews, and Zou (2019) for a more detailed discussion of the low reserve price puzzle.



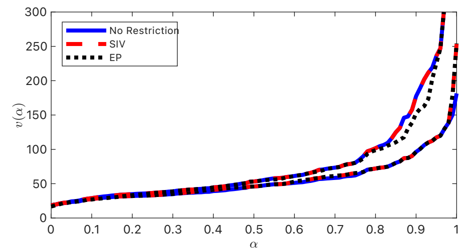
(a) $n = 2$



(b) $n = 3$

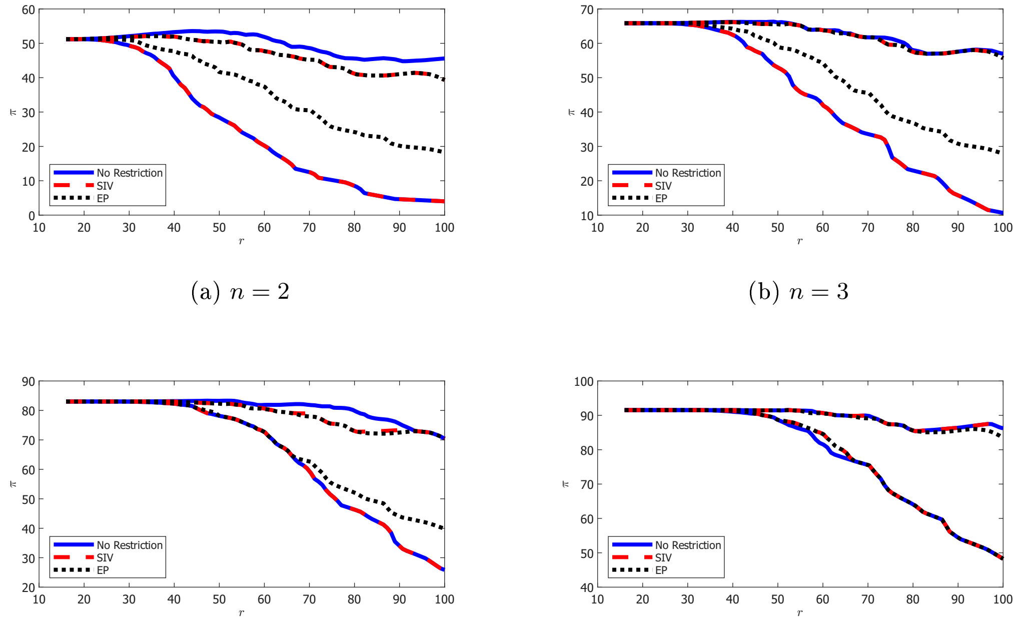


(c) $n = 4$



(d) $n = 5$

Figure 7: **Bounding the Valuation Distribution (Timber Data):** Bounds for the valuation quantile function using RNBR overbidding in distribution. Each figure shows bounds without using variation in the number of bidders as a solid blue line, bounds that use the SIV assumption (Assumption 7) as a red dashed line, and the bounds using the EP assumption (Assumption 6) as a black dotted line.



(a) $n = 2$ (b) $n = 3$

(c) $n = 4$ (d) $n = 5$

Figure 8: **Bounding the Seller's Profit (Timber Data):** Bounds for the seller's profit using RNBR overbidding in distribution. Each figure shows bounds without using variation in the number of bidders as a solid blue line, bounds that use the SIV assumption (Assumption 7) as a red dashed line, and the bounds using the EP assumption (Assumption 6) as a black dotted line.

7 Extensions: Heterogeneous Bidders

7.1 Unknown Bidder Types

Although our bounds for the valuation quantile are robust to asymmetric bidding strategies, the result for the profit function requires homogeneity. This section considers an extension where bidders use heterogeneous bidding strategies but are unaware of this heterogeneity.

Suppose there are M types of bidders and they have the same valuation distribution F_n but the bidding strategies depend on the types. The fraction and bid distribution of bidder type i are ω_i and $G_{n,i}$, respectively. Denote the density of $G_{n,i}$ by $g_{n,i}$. Then the observed bid distribution can be written as

$$G_n(b) = \sum_{i=1}^{M} \omega_i G_{n,i}(b).$$

Assumption 9. *A type i bidder bids according to Assumption 5 with $\lambda_n = \lambda_{n,i}$ and $H_n = H_{n,i}$ for $i = 1, \cdots, M$.*

Under Assumption 9, a bidder with type i overbids compared to the RNBR of the unobserved bid distribution $G_{n,i}$ but does not necessarily overbid compared to the RNBR of the observed bid distribution G_n.

The econometrician observes G_n and knows the maximum number of types $\bar{M} < \infty$. The goal is to obtain bounds for valuation quantile function and the counterfactual profits. With a slight abuse of notation denote the inverse of the RNBR to a bid distribution G with a density g in an n-bidder auction by

$$\rho_n^{-1}(\alpha, G) = G^{-1}(\alpha) + \frac{\alpha}{(n-1)\, g\left(G^{-1}(\alpha)\right)}.$$

Then the upper bound on the valuation quantile function is

$$\begin{aligned} &\bar{v}_n^H(\alpha) = \max_{\tilde{G}_{n,i}, \tilde{\omega}_i} \min_{i=1,\cdots,\bar{M}} \rho_n^{-1}\left(\alpha, \tilde{G}_{n,i}\right), \\ &\text{st. } G_n(b) = \sum_{i=1}^{\bar{M}} \tilde{\omega}_i \tilde{G}_{n,i}(b), \sum_{i=1}^{\bar{M}} \tilde{\omega}_i = 1, \text{ and } \tilde{\omega}_i \geq 0 \text{ for all } i = 1, \cdots, \bar{M}, \text{ and } b. \end{aligned}$$

The idea is the following. First, suppose we know the bid distribution of each bidder type, $\tilde{G}_{n,i}$, $i = 1, \cdots, M$. For each bidder type i, we can then apply the method developed in Section 4 to $\tilde{G}_{n,i}(b)$ and obtain an upper bound on the valuation quantile function for each i. Because the valuation quantile is the same for every bidder type, its upper bound can then be obtained by taking the minimum of the upper bounds across all bidder types. Next, because we do not know the bid distribution and the fraction of each bidder type, we need to consider all possibilities that are consistent with the observed bid distribution. To this end, we search over all possible weights and bid distributions of every bidder type that are consistent with G_n to find the maximum valuation quantile. This leads to our final upper bound on the valuation quantile function. Notice that because we do not know the total number of types M, we replace it with $\bar{M}$, which is a known upper bound of M. The lower bound for the valuation quantile function is again the bid quantile function: $\underline{v}_n^H(\alpha) = b_n(\alpha)$.

To obtain the upper bound for counterfactual profit, we again first calculate the upper bound for the counterfactual bid quantile function given $\tilde{G}_{n,i}$ and $\bar{v}_n^H$ following the method described in Section 4. With a little abuse of notation, let $\sigma_n\left(\alpha, r, \tilde{G}_{n,i}\right)$ be the α-th quantile under RNBNE when the bid distribution without a reserve price is $\tilde{G}_{n,i}$. Define

$$\phi_n\left(\alpha, r, \tilde{G}_{n,i}\right) = \min\left\{\sigma_n\left(\alpha, r, \tilde{G}_{n,i}\right), \bar{v}_n^H(\alpha)\right\}$$

and the lower bound for the bid distribution under r as

$$\underline{\Psi}_n\left(b, r, \tilde{G}_{n,i}\right) = \inf\left\{\alpha : \phi\left(\alpha, r, \tilde{G}_{n,i}\right) \geq b\right\}.$$

Then the counterfactual bid distribution is a mixture of $\underline{\Psi}_n\left(b, r, \tilde{G}_{n,i}\right)$ and the profit function can be obtained by

$$\bar{\pi}_n^H(r) = \max_{\tilde{G}_{n,i},\tilde{\omega}_i} \int_r^{\infty} (x-c)\, d\left(\sum_{i=1}^{\bar{M}} \tilde{\omega}_i \underline{\Psi}_n\left(x, r, \tilde{G}_{n,i}\right)\right)^n + c$$

$$\text{st. } G_n(b) = \sum_{m=1}^{\bar{M}} \tilde{\omega}_i \tilde{G}_{n,i}(b), \sum_{i=1}^{\bar{M}} \tilde{\omega}_i = 1, \text{ and } \tilde{\omega}_i \geq 0 \text{ for all } i = 1, \cdots, \bar{M}, \text{ and } b.$$

Again, the lower profit bound, denoted by $\underline{\pi}_n^H$ is obtained by assuming that every bidder is bidding truthfully. It is the same as $\underline{\pi}_n$ defined by (8).

Proposition 10. *Suppose only $G_n(b)$ is observed and Assumption 9 holds, then $\underline{v}_n^H(\alpha) \leq v_n(\alpha) \leq \bar{v}_n^H(\alpha)$ and $\underline{\pi}_n^H(r) \leq \pi_n(r) \leq \bar{\pi}_n^H(r)$.*

To apply the identification results, we can specify a large $\bar{M}$ and approximate $\tilde{G}_{n,i}$ using sieves. Then we can impose the constraint $G_n(b) = \sum_{m=1}^{\bar{M}} \tilde{\omega}_i \tilde{G}_{n,i}(b)$ on a grid of b.

The method proposed above is feasible but can be computationally challenging if $\bar{M}$ is large. However, in many data sets, we observe a bidder's bids in multiple auctions, which allows us to use the techniques from the measurement error literature to identify the number of types, the fraction of each type and the bid distribution of each type. An (2017) first exploited this idea to identify heterogeneous beliefs.[24] Here we adopt a version of the results to our problem.

Let $\boldsymbol{G}_n(\cdot,\cdot)$ be the joint distribution of a bidder's bids in two different n-bidder auctions. A bidder's type is i with probability ω_i for $i = 1, \cdots, M$ and types are permanent, but bidders draw new valuations in each auction independently. We impose the following assumption.

Assumption 10. *The highest bid for each type, $b_{n,i}(1)$, differs across i and $g_{n,i}$ is continuous and bounded away from 0 on its support for $i = 1, \cdots, M$.*

This assumption allows us to apply Theorem 1 in Hu and Sasaki (2017) to obtain ω_i and $G_{n,i}$ for $i = 1, \cdots, M$. Then we can define

$$\bar{v}_n^R(\alpha) = \min_{i=1,\cdots,M} \rho_n^{-1}(\alpha, G_{n,i}),\ \underline{v}_n^R(\alpha) = \max_{i=1,\cdots,M} b_{n,i}(\alpha).$$

Moreover, the upper bound of the profit can be obtained by

$$\bar{\pi}_n^R(r) = \int_r^{\infty} (x-c)\, d\left(\sum_{i=1}^{M} \omega_i \underline{\Psi}_n(x, r, G_{n,i})\right)^n + c.$$

[24]This strategy requires that the bid distributions from different bidder types can be ranked in some way. A typical assumption is that they are ranked in the sense of first-order stochastic dominance, which can be restrictive in many applications. Luo (2018) shows that requiring a local first-order stochastic dominance ranking is sufficient and is satisfied in first-price auction models under weak assumptions. Other applications of measurement error methods to auctions include McAdams, Hu, and Shum (2010), Krasnokutskaya (2011) and Li, Perrigne, and Vuong (2000), among others.

To obtain a lower profit bound, define $\underline{\sigma}_n^R(\alpha, r)$ to be the RNBNE bid quantile under $\underline{v}_n^R$ and reserve price r. Define an upper bound for the bid distribution for type i as

$$\bar{\Psi}_{n,i}(b, r) = \inf\left\{\alpha : \max\left\{\underline{\sigma}_n^R(\alpha, r), b_{n,i}(\alpha)\right\} \geq b\right\}.$$

Then the lower profit bound is

$$\underline{\pi}_n^R(r) = \int_r^{\infty} (x - c)\, d\left(\sum_{i=1}^{M} \omega_i \bar{\Psi}_{n,i}(x, r)\right)^n + c.$$

Proposition 11. *If Assumption 10 holds, then ω_i and $G_{n,i}(\cdot)$ for $i = 1, \cdots, M$ are identified from $\boldsymbol{G}_n(\cdot,\cdot)$. Moreover, if Assumption 9 holds, $\underline{v}_n^R(\alpha) \leq v_n(\alpha) \leq \bar{v}_n^R(\alpha)$ and $\underline{\pi}_n^R(r) \leq \pi_n(r) \leq \bar{\pi}_n^R(r)$.*

7.2 Known Bidder Types

So far we have studied the case where the bidder heterogeneity is unknown to the rival bidders. This section considers the case where this bidder heterogeneity is known and common knowledge.

Suppose that bidder type i's bid distribution is $G_{n,i}$ for $i = 1, 2, \cdots, n$ in an auction with n bidders. Notice that here we allow each bidder to have a different type. Here $G_{n,i}$ is the bid distribution of bidders of the same type as bidder i in auctions with the same bidder type composition. The best response of bidder i is

$$\rho_{n,i}(t) = \max_b (t - b)\, \mathbb{P}\left(\max_{j \neq i} b_j \leq b\right) = \max_b (t - b)\, \Pi_{j \neq i} G_{n,j}(b),$$

which implies that

$$\rho_{n,i}^{-1}(b) = b + \frac{1}{\sum_{j \neq i} \frac{g_{n,j}(b)}{G_{n,j}(b)}}.$$

Let $s_{n,i}$ be the bid strategy of bidder i. The next assumption is an analogue of Assumption 2.

Assumption 11. *Bidders overbid compared to the RNBR bidding strategy: $s_{n,i}(\cdot) \geq \rho_{n,i}(\cdot)$ for all i.*

This assumption is stronger than required for the results in this section. It is sufficient that bidders overbid with respect to RNBR in distribution such that $F_n\left(\rho_{n,i}^{-1}(b)\right) \geq G_{n.i}(b)$, which is implied by Assumption 11. Such a weaker restriction could be relevant in a model with an additional level of heterogeneity where bidding behavior conditional on the valuation is not fully determined by the type i.

Proposition 12. *When bidders' types i are common knowledge, first-price auction models with risk aversion, loser regret, Choquet expected utility or loss aversion in money satisfy Assumption 11.*

We can then apply the same method as in the case with unknown bidder types studied above. The only difference is that we can get a bound for each type, which can help to tighten the bounds for the valuation distribution. We can then define

$$\underline{v}_n(\alpha) = \max_{i=1,2,\cdots,n} G_{n,i}^{-1}(\alpha), \tag{12}$$

$$\bar{v}_n(\alpha) = \min_{i=1,2,\cdots,n} \rho_{n,i}^{-1}\left(G_{n,i}^{-1}(\alpha)\right). \tag{13}$$

Proposition 13. *Suppose bidder types i are common knowledge. If the bidders are rational (Assumption 4) and overbid compared to RNBR (Assumption 11), then $\underline{v}_n(\alpha) \leq v_n(\alpha) \leq \bar{v}_n(\alpha)$, where $\underline{v}_n(\alpha)$ and $\bar{v}_n(\alpha)$ are defined by (12) and (13), respectively.*

8 Conclusion

We show that the canonical approach for empirical analysis of first-price auctions developed in Guerre, Perrigne, and Vuong (2000) can be reinterpreted as a bound even if the identifying assumption that bidders play the RNBNE strategy fails. The key assumption that enables this reinterpretation is that bidders overbid compared to RNBR in distribution. We show that this assumption is consistent with many alternative models proposed in the literature and with experimental bid data. The resulting bounds are tighter than the ones obtained under RNBNE overbidding. Practitioners can therefore continue to use the method developed in Guerre, Perrigne, and Vuong (2000) even if they are concerned that bidders deviate from the RNBNE strategy.

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A Online Appendix

Proof of Proposition 3. Let s_n be the bidding strategy that rationalizes G_n with valuation distribution F_n. Then

$$s_n^{-1}(b) = F_n^{-1}(G_n(b)) = F_n^{-1}([pG_{n,1} + (1-p)G_{n,2}](b)).$$

The RNBR to G_n satisfies

$$\rho_n^{-1}(b) = b + \frac{1}{n-1}\frac{pG_{n,1}(b) + (1-p)G_{n,2}(b)}{pg_{n,1}(b) + (1-p)g_{n,2}(b)}.$$

Notice that if $s_n(t) \geq \rho_n(t)$ for all t if and only if $s_n^{-1}(b) \leq \rho_n^{-1}(b)$ for all b. We the latter holds for all sufficiently large p by analyzing the limit as $p \to 1$. Heuristically, $s_n^{-1}(b) \to s_{n,1}^{-1}(b)$ and $\rho_n^{-1}(b) \to \rho_{n,1}^{-1}(b)$ as $p \to 1$. Because $s_{n,1}^{-1}(b) < \rho_{n,1}^{-1}(b)$ for $b > 0$ by assumption, one can show that for all sufficiently large p, $s_n^{-1}(b) \leq \rho_n^{-1}(b)$ for all b that are away from 0. The challenge is to show this same holds in the neighborhood of 0. For that, we exploit the converges in the derivatives of $s_n^{-1}(b)$ and $\rho_n^{-1}(b)$. The following is the formal proof.

Notice $F_n\left(s_{n.1}^{-1}(b)\right) = G_{n,1}(b)$. Take derivative with respect to b on both sides and rearrange to obtain

$$\frac{ds_{n,1}^{-1}(b)}{db} = \frac{g_{n,1}(b)}{f_n\left(F_n^{-1}(G_{n,1}(b))\right)}.$$

The definition of $\rho_{n,1}^{-1}$ implies that

$$\frac{d\rho_{n,1}^{-1}(b)}{db} = 1 + \frac{1}{n-1} - \frac{G_{n,1}(b)\,g_{n,1}'(b)}{g_{n,1}(b)^2}.$$

Because $G_{n,1}$ has a continuous second-order derivative and f_n is continuous and bounded away from 0 on its support, both derivatives are continuous in b and so is

$$D(b) = \frac{d\rho_{n,1}^{-1}(b)}{db} - \frac{ds_{n,1}^{-1}(b)}{db}.$$

Moreover, $s_{n,1}^{-1}(b) < \rho_{n,1}^{-1}(b)$ for all $b > 0$ and the fact $s_{n,1}^{-1}(0) = \rho_{n,1}^{-1}(0) = 0$ imply that $D(0) > 0$. By the continuity of D, there exist $\varepsilon > 0$ and $\gamma > 0$ such that $D(b) > \gamma$ for all $b \in [0, \varepsilon]$. Next notice that $\rho_{n,1}^{-1}(b) - s_{n,1}^{-1}(b)$ is continuous in b. Therefore, $\rho_{n,1}^{-1}(b) - s_{n,1}^{-1}(b) > 0$ for all $b > 0$ in the support of $G_{n,1}$ implies there exists η such that $\rho_{n,1}^{-1}(b) - s_{n,1}^{-1}(b) > \eta$ for all $b > \varepsilon$ in the support of $G_{n,1}$.

Because $G_{n,1}$, $G_{n,2}$ are bounded and f_n is bounded away from 0 on its support,

$$\frac{ds_n^{-1}(b)}{dp} = \frac{G_{n,1}(b) - G_{n,2}(b)}{f_n\left(F_n^{-1}([pG_{n,1} + (1-p)G_{n,2}](b))\right)}.$$

is bounded uniformly in b. As a result, $s_n^{-1}(b)$ converges uniformly to $s_{n,1}^{-1}$ as $p \to 1$.

The RNBR strategy under G_n is

$$\rho_n^{-1}(b) = b + \frac{1}{n-1} \frac{pG_{n,1}(b) + (1-p) G_{n,2}(b)}{pg_{n,1}(b) + (1-p) g_{n,2}(b)}.$$

Its derivative with respect to p is bounded uniformly in b. Therefore, it converges uniformly to $\rho_{n,1}^{-1}(b)$ as $p \to 1$. Because $\rho_{n,1}^{-1}(b) > s_{n,1}^{-1}(b) + \eta$ for all $b > \varepsilon$ in the support of $G_{n,1}$, there exists $\bar{p}_1 < 1$ such that if $p > \bar{p}_1$, $\rho_n^{-1}(b) > s_n^{-1}(b)$for all $b \geq \varepsilon$ and in the support of $G_{n,1}$, i.e., for all b that are away from 0.

Next we show that the same holds in the neighborhood of 0 by investigating the derivatives. Notice that

$$\frac{ds_n^{-1}(b)}{db} = \frac{pg_{n,1}(b) + (1-p) g_{n,2}(b)}{f_n\left(F_n^{-1}\left([pG_{n,1} + (1-p) G_{n,2}](b)\right)\right)}.$$

Its derivative with respect to p is bounded uniformly in b because $g_{n,1}$ and $g_{n,2}$ are bounded and f_n' is continuous on the support of f_n and hence bounded. This implies that $ds_n^{-1}(b)/db$ converges to $ds_{n,1}^{-1}(b)/db$ uniformly in b as p converges to 1. Similarly,

$$\frac{d\rho_n^{-1}(b)}{db} = 1 + \frac{1}{n-1} - \frac{[pG_{n,1}(b) + (1-p) G_{n,2}(b)]\left[pg_{n,1}'(b) + (1-p) g_{n,2}'(b)\right]}{[pg_{n,1}(b) + (1-p) g_{n,2}(b)]^2}.$$

converges to $d\rho_{n,1}^{-1}(b)/db$ uniformly as p converges to 1. This implies that there exists $\bar{p}_2 < 1$ such that

$$\frac{ds_n^{-1}(b)}{db} < \frac{d\rho_n^{-1}(b)}{db}$$

for all $p > \bar{p}_2$, which in turn implies that $s_n^{-1}(b) > \rho_n^{-1}(b)$ for all $b \leq \varepsilon$. The proposition follows with $\bar{p} = \max(\bar{p}_1, \bar{p}_2)$. □

We omit the proofs of Propositions 6 and 7 because they are straightforward.

Lemma 2. *Let $b_n(\alpha, r)$ be the bid quantile function under the valuation quantile $v_n(\cdot)$ and reserve price r. Let $\beta_{n,G}(\alpha, r)$ be the RNBNE bid quantile under the valuation quantile $v_{n,G}(\cdot)$ and the reserve price r. Then if the bid function of $b_n(\cdot, r)$ satisfies Assumption 5 and $b_n(\alpha, 0) = \beta_{n,G}(\alpha, 0)$ for all $\alpha \in [0,1]$, then $b_n(\alpha, r) \leq \beta_{n,G}(\alpha, r)$ for all $\alpha \in [0,1]$ and $r > 0$.*

Proof. Because $b_n(\alpha, 0) = \beta_{n,G}(\alpha, 0)$ for all $\alpha \in [0,1]$, $\partial b_n(\alpha, 0)/\partial\alpha = \partial\beta_{n,G}(\alpha, 0)/\partial\alpha$ for all α, which implies that if $\alpha > 0$

$$\lambda_n(v_n(\alpha) - b_n(\alpha, 0)) \mathcal{H}_n(\alpha) = v_{n,G}(\alpha) - b_n(\alpha, 0).$$

Because $\mathcal{H}_n(\cdot) = H_n(v_n(\cdot)) \geq 1$ and $\lambda_n'(\cdot) \geq 1$, we must have $v_n(\alpha) \leq v_{n,G}(\alpha)$ for all α. Let α_r and $\bar{\alpha}_r$ satisfy $v_n(\alpha_r) = r$ and $v_{n,G}(\bar{\alpha}_r) = r$. Obviously, $\bar{\alpha}_r \leq \alpha_r$. Because $b_n(\alpha, r) = 0$ if $\alpha < \alpha_r$ and $\beta_{n,G}(\alpha_r, r) \geq \beta_{n,G}(\bar{\alpha}_r, r) = r = b_n(\alpha_r, r)$, $\beta_{n,G}(\alpha, r) \geq b_n(\alpha, r)$ for all $\alpha \leq \alpha_r$. We next show by contradiction that $\beta_{n,G}(\alpha,r) \geq b_n(\alpha,r)$, $\forall\alpha \in [\alpha_r,1]$. Suppose there is an $\alpha_1 \in [\alpha_r,1]$ such that $\beta_{n,G}(\alpha_1,r) < b_n(\alpha_1,r)$. Define

$$\bar{\alpha} = \sup\{\alpha \in [\alpha_r,\alpha_1] : \beta_{n,G}(\alpha,r) \geq b_n(\alpha,r)\}.$$

By continuity of the bid function and the fact that $\beta_{n,G}(\alpha_r,r) \geq b_n(\alpha_r,r)$, we must have $\beta_{n,G}(\bar{\alpha},r) = b_n(\bar{\alpha},r)$ and there exists a $\Delta > 0$ such that for any $\alpha \in (\bar{\alpha},\bar{\alpha}+\Delta)$, $\beta_{n,G}(\alpha,r) < b_n(\alpha,r)$. This means that there exists at least some $\alpha \in [\bar{\alpha},\bar{\alpha}+\Delta]$ such that $\partial\beta_{n,G}(\alpha,r)/\partial\alpha < \partial b_n(\alpha,r)/\partial\alpha$ or equivalently,

$$\lambda_n(v_n(\alpha) - b_n(\alpha,r))\mathcal{H}_n(\alpha) > v_{n,G}(\alpha) - \beta_{n,G}(\alpha,r).$$

Because $b_n(\alpha,r) - b_n(\alpha,0) \geq 0$, $\lambda_n' \geq 1$ and $\mathcal{H}_n \geq 1$,

$$\begin{aligned}
\lambda_n(v_n(\alpha) - b_n(\alpha,0))\mathcal{H}_n(\alpha) &\geq \lambda_n(v_n(\alpha) - b_n(\alpha,r))\mathcal{H}_n(\alpha) + b_n(\alpha,r) - b_n(\alpha,0) \\
&> v_{n,G}(\alpha) - \beta_{n,G}(\alpha,r) + b_n(\alpha,r) - b_n(\alpha,0) \\
&> v_{n,G}(\alpha) - b_n(\alpha,0),
\end{aligned}$$

which violates the fact that

$$(n-1)\lambda_n(v_n(\alpha) - b_n(\alpha))\mathcal{H}_n(\alpha) = (n-1)[v_{n,G}(\alpha) - b_n(\alpha)]$$

for all $\alpha \in (0,1]$. Thus, $\beta_{n,G}(\alpha,r) \geq b_n(\alpha,r)$, $\forall\alpha \in [\alpha_r,1]$. $\square$

Proof of Proposition 1. The first two results are straightforward and we only prove the last claim. If $s_n(t) \geq \rho_n(t)$, by definition of ρ_n

$$t \leq s_n(t) + \frac{1}{n-1}\frac{G_n(s_n(t))}{g_n(s_n(t))}.$$

Because $g_n(s_n(t)) = f_n(s_n(t))/s_n'(t)$ and $G_n(s_n(t)) = F_n(t)$. Then we can rearrange the above inequality as

$$s_n'(t) \geq [t - s_n(t)](n-1)\frac{F_n(t)}{f_n(t)}. \tag{14}$$

Recall that $s_n(0) = 0$, $\sigma_n(0) = 0$ and

$$\sigma_n'(t) = [t - \sigma_n(t)](n-1)\frac{F_n(t)}{f_n(t)}. \tag{15}$$

Now towards contradiction, suppose there exists $\tilde{t} > 0$ such that $s_n(\tilde{t}) < \sigma_n(\tilde{t})$. Let

$$\bar{t} = \sup\{t \leq \tilde{t} : \sigma_n(t) = s_n(t)\}.$$

Because $s_n(0) = \sigma_n(0) = 0$, $\bar{t}$ is well-defined. Because the bidding strategies are continuous, $\bar{t} < \tilde{t}$, $s_n(\bar{t}) = \sigma_n(\bar{t})$ and $s_n(t) < \sigma_n(t)$ if $t \in (\bar{t},\tilde{t}]$. This combined with (14)-(15) implies that

$s_n'(t) > \sigma_n'(t)$ on $(\bar{t}, \tilde{t}]$. Therefore,

$$s_n(\bar{t}) = s_n(\tilde{t}) - \int_{\bar{t}}^{\tilde{t}} s_n'(x)\,dx < \sigma_n(\tilde{t}) - \int_{\bar{t}}^{\tilde{t}} \sigma_n'(x) = \sigma_n(\bar{t}).$$

This contradicts $s_n(\bar{t}) = \sigma_n(\bar{t})$ and the last claim of this proposition follows. $\square$

Proof of Proposition 9. We only prove the consistency results for $\hat{\bar{v}}_n$ and $\hat{\bar{\pi}}_n$. Other results can be shown in a similar fashion.

We start with proving $\hat{\bar{v}}_n$ that converges in probability to $\bar{v}_n$ uniformly on $A_n(\varepsilon)$ for any sufficiently small $\varepsilon > 0$. Define

$$\underline{F}_n(v) = \mathbb{P}\left(\rho_n^{-1}(y_{i,n}) \leq v\right).$$

Notice that $\underline{F}_n^{-1}(\alpha) = \bar{v}_n(\alpha)$ if $\alpha \in A_n(\varepsilon)$. To see this, recall that by definition $\bar{v}_n(\alpha) = \rho_n^{-1}(b_n(\alpha))$. Therefore

$$\begin{aligned}
\underline{F}_n(\bar{v}_n(\alpha)) &= \mathbb{P}\left(\rho_n^{-1}(y_{i,n}) \leq \bar{v}_n(\alpha)\right) = \mathbb{P}\left(\rho_n^{-1}(y_{i,n}) \leq \rho_n^{-1}(b_n(\alpha))\right) \\
&= \mathbb{P}(y_{i,n} \leq b_n(\alpha)) = \alpha.
\end{aligned}$$

The second to last equality holds because ρ_n^{-1} is strictly increasing. The last equality holds because G_n is continuous. Moreover, if $\alpha \in A_n(\varepsilon)$, $\bar{v}_n(\cdot)$ is continuous at α, which implies that $\bar{v}_n(\alpha)$ is the unique solution to $\underline{F}_n(x) = \alpha$. To see this, suppose towards a contradiction, there exists $\tilde{v} \neq \bar{v}_n(\alpha)$ such that $\underline{F}_n(\tilde{v}) = \alpha$. Because $\bar{v}_n(\cdot)$ is continuous at α, there exists $\delta > 0$ such that $\tilde{v} \notin [\bar{v}_n(\alpha - \delta), \bar{v}_n(\alpha + \delta)]$. Then either

$$\underline{F}_n(\tilde{v}) \leq \underline{F}_n(\bar{v}_n(\alpha - \delta)) = \alpha - \delta$$

or

$$\underline{F}_n(\tilde{v}) > \underline{F}_n(\bar{v}_n(\alpha + \delta)) = \alpha + \delta.$$

We arrive at a contradiction. Therefore, $\bar{v}_n(\alpha)$ is the unique solution to $\underline{F}_n(v) = \alpha$, which implies $\underline{F}_n^{-1}(\alpha) = \bar{v}_n(\alpha)$.

We next show that

$$\sup_{v \in [\bar{v}_n(0), \bar{v}_n(1)]} \left| \hat{\underline{F}}_n(v) - \underline{F}_n(v) \right| \to^p 0.$$

Notice that by Assumption 8,

$$\sup_{b \in B_n(\varepsilon)} \left| \widehat{\rho_n^{-1}}(b) - \rho_n^{-1}(b) \right| \to^p 0 \tag{16}$$

for every $\varepsilon > 0$. We can rewrite $\hat{\underline{F}}_n(v)$ as

$$\hat{\underline{F}}_n(v) = \frac{1}{N_n}\sum_{i=1}^{N_n} \mathbf{1}\left(\widehat{\rho_n^{-1}}(y_{i,n}) \le v, y_{i,n} \in B_n(\varepsilon)\right) + \frac{1}{N_n}\sum_{i=1}^{N_n} \mathbf{1}\left(\widehat{\rho_n^{-1}}(y_{i,n}) \le v, y_{i,n} \notin B_n(\varepsilon)\right).$$

This implies that

$$D_{1,N} \le \hat{\underline{F}}_n(v) \le D_{2,N}$$

where

$$\begin{aligned} D_{1,N} &= \frac{1}{N_n}\sum_{i=1}^{N_n} \mathbf{1}\left(\widehat{\rho_n^{-1}}(y_{i,n}) \le v, y_{i,n} \in B_n(\varepsilon)\right), \\ D_{2,N} &= \frac{1}{N_n}\sum_{i=1}^{N_n} \mathbf{1}\left(\widehat{\rho_n^{-1}}(y_{i,n}) \le v, y_{i,n} \in B_n(\varepsilon)\right) + \frac{1}{N_n}\sum_{i=1}^{N_n} \mathbf{1}\left(y_{i,n} \notin B_n(\varepsilon)\right). \end{aligned}$$

Because of (16), uniformly in v,

$$\begin{aligned} D_{1,N} &\ge \frac{1}{N_n}\sum_{i=1}^{N_n} \mathbf{1}\left(\rho_n^{-1}(y_{i,n}) \le v-\delta, y_{i,n} \in B_n(\varepsilon)\right) - \frac{1}{N_n}\sum_{i=1}^{N_n} \mathbf{1}\left(\left|\widehat{\rho_n^{-1}}(y_{i,n}) - \rho_n^{-1}(y_{i,n})\right| > \delta, y_{i,n} \in B_n(\varepsilon)\right) \\ &= \frac{1}{N_n}\sum_{i=1}^{N_n} \mathbf{1}\left(\rho_n^{-1}(y_{i,n}) \le v-\delta, y_{i,n} \in B_n(\varepsilon)\right) - o_p(1) \\ &\ge \frac{1}{N_n}\sum_{i=1}^{N_n} \left[\mathbf{1}\left(\rho_n^{-1}(y_{i,n}) \le v-\delta\right) - \mathbf{1}\left(y_{i,n} \notin B_n(\varepsilon)\right)\right] - o_p(1). \end{aligned}$$

By definition, $\underline{F}_n(\cdot)$ is the CDF of $\rho_n^{-1}(y_{i,n})$. Therefore, $D_{1,N}$ converges uniformly in v to $\underline{F}_n(v-\delta) - \mathbb{E}\mathbf{1}(y_{i,n} \notin B_n(\varepsilon))$ in probability for any $\delta > 0$ and $\varepsilon > 0$. Similarly, uniformly in v,

$$D_{2,N} \le \frac{1}{N_n}\sum_{i=1}^{N_n} \mathbf{1}\left(\rho_n^{-1}(y_{i,n}) \le v+\delta\right) + \frac{1}{N_n}\sum_{i=1}^{N_n} \mathbf{1}\left(y_{i,n} \notin B_n(\varepsilon)\right) + o_p(1)$$

which converges uniformly in v to $\underline{F}_n(v+\delta) + \mathbb{E}\mathbf{1}(y_{i,n} \notin B_n(\varepsilon))$ in probability. This implies that given any $\delta, \varepsilon > 0$,

$$\hat{\underline{F}}_n(v) \in \left[\underline{F}_n(v-\delta) - \mathbb{E}\mathbf{1}(y_{i,n} \notin B_n(\varepsilon)), \underline{F}_n(v+\delta) + \mathbb{E}\mathbf{1}(y_{i,n} \notin B_n(\varepsilon))\right]$$

for all $v \in [\bar{v}_n(0), \bar{v}_n(1)]$ with a probability approaching 1 as $N \to \infty$. Because $\rho_n^{-1}(y_{i,n})$ is strictly increasing, $\underline{F}_n$ does not have mass points and is hence continuous. Therefore, it is uniformly continuous on the compact set $[\bar{v}_n(0) - \delta, \bar{v}_n(1) + \delta]$. As $\delta \downarrow 0$, $\underline{F}_n(v-\delta) \to \underline{F}_n(v)$ and $\underline{F}_n(v+\delta) \to \underline{F}_n(v)$ uniformly in $v \in [\bar{v}_n(0), \bar{v}_n(1)]$. Moreover as $\varepsilon \downarrow 0$, $\mathbb{E}\mathbf{1}(y_{i,n} \notin B_n(\varepsilon)) \to 0$.

Therefore,

$$\sup_{v\in[\bar{v}_n(0),\bar{v}_n(1)]}\left|\hat{\underline{F}}_n(v)-\underline{F}_n(v)\right|\to^p 0.$$

This implies that if $\underline{F}_n$ is continuous and strictly increasing on $[a,b]$, $\hat{\underline{F}}_n^{-1}$ converges to $\underline{F}_n^{-1}$ uniformly on $[\underline{F}_n(a),\underline{F}_n(b)]$ in probability. Because $\underline{F}_n(\cdot)$ is strictly increasing and continuous on $\left[\underline{F}_n^{-1}(0),\underline{F}_n^{-1}(\alpha_{n,1}-\varepsilon)\right]$, $\left[\underline{F}_n^{-1}(\alpha_{n,L}+\varepsilon),\underline{F}_n^{-1}(1)\right]$ and $\left[\underline{F}_n^{-1}(\alpha_{n,l-1}+\varepsilon),\underline{F}_n^{-1}(\alpha_{n,l}-\varepsilon)\right]$ for all $l=1,2,\cdots,L$. Then $\hat{\underline{F}}_n^{-1}$ converges uniformly to $\underline{F}_n^{-1}$ in probability on each of $[0,\alpha_{n,1}-\varepsilon]$, $[\alpha_{n,L}+\varepsilon,1]$ and $[\alpha_{n,l-1}+\varepsilon,\alpha_{n,l}-\varepsilon]$, $l=2,3,\cdots,L$. Therefore, $\hat{\underline{F}}_n^{-1}$ converges in probability to $\underline{F}_n^{-1}$ uniformly on $A_n(\varepsilon)$ for any $\varepsilon>0$. Because $\underline{F}_n^{-1}(\alpha)=\bar{v}_n(\alpha)$ if $\alpha\in A_n(\varepsilon)$ and by definition $\hat{\underline{F}}_n^{-1}=\hat{\bar{v}}_n$, the uniform convergence of $\hat{\bar{v}}_n$ to $\bar{v}_n$ on $A_n(\varepsilon)$ in probability follows.

Next, we prove the convergence of $\hat{\bar{\pi}}_n(r)$ to $\bar{\pi}_n(r)$. We start with proving that given $r\geq 0$, $\hat{\bar{\sigma}}_n(\alpha,r)\to^p\bar{\sigma}_n(\alpha,r)$ for almost every $\alpha\in(0,1]$. Notice that uniform convergence of $\hat{\bar{v}}_n$ to $\bar{v}_n$ on $A_n(\varepsilon)$ for every $\varepsilon>0$ implies that for almost every $\alpha\in[0,1]$, $\hat{\bar{v}}_n(\alpha)$ converges to $\bar{v}_n(\alpha)$ in probability. If $\bar{v}_n(\alpha)<r$, then $\hat{\bar{v}}_n(\alpha)<r$ with probability approaching 1 as $N\to\infty$, which implies that $\hat{\bar{\sigma}}_n(\alpha,r)=0=\bar{\sigma}_n(\alpha,r)$ with probability approaching 1. If $\bar{v}_n(\alpha)>r$, then $\hat{\bar{v}}_n(\alpha)>r$ with probability approaching 1 as $N\to\infty$. Then

$$\begin{aligned}
&\alpha^{n-1}\left|\hat{\bar{\sigma}}_n(\alpha,r)-\bar{\sigma}_n(\alpha,r)\right|\\
=&\left|r\hat{\bar{v}}_n^{-1}(r)^{n-1}+(n-1)\int_{r\leq\hat{\bar{v}}_n(x)}^{\alpha}x^{n-2}\hat{\bar{v}}_n(x)\,dx-r\bar{v}_n^{-1}(r)^{n-1}-(n-1)\int_{r\leq\bar{v}_n(x)}^{\alpha}x^{n-2}\bar{v}_n(x)\,dx\right|\\
\leq& r\left|\hat{\bar{v}}_n^{-1}(r)^{n-1}-\bar{v}_n^{-1}(r)^{n-1}\right|+(n-1)\left|\int_{r\leq\hat{\bar{v}}_n(x)}^{\alpha}x^{n-2}\hat{\bar{v}}_n(x)\,dx-\int_{r\leq\bar{v}_n(x)}^{\alpha}x^{n-2}\bar{v}(x)\,dx\right|. \qquad (17)
\end{aligned}$$

By Assumption 8, $\hat{\bar{v}}_n(x)\mathbf{1}\left(r\leq\hat{\bar{v}}_n(x)\right)$ converges to $\bar{v}_n(x)\mathbf{1}(r\leq\bar{v}_n(x))$ in probability for almost every x. Therefore, by the Dominated Convergence theorem, the second term of (17) converges to 0 in probability. It remains to show the first term converges to 0 in probability. Notice that if $r\in(\bar{v}_n(0),\bar{v}_n(1))$, there exists a unique α^r such that $r\in\left[\bar{v}_n(\alpha^r),\bar{v}_n(\alpha^r)_+\right]$, where $\bar{v}_n(\alpha^r)_+=\lim_{x\downarrow\alpha^r}\bar{v}_n(x)$. This implies that $\bar{v}_n^{-1}(r)=\alpha^r$. Because $\hat{\bar{v}}_n(\alpha)$ converges to $\bar{v}_n(\alpha)$ in probability for almost every $\alpha\in[0,1]$, there exists a sequence $\zeta_k>0$ such that $\zeta_k\to 0$ and for every ζ_k, as $N\to\infty$

$$\begin{aligned}
\hat{\bar{v}}_n(\alpha^r-\zeta_k)&\to^p\bar{v}_n(\alpha^r-\zeta_k),\\
\hat{\bar{v}}_n(\alpha^r+\zeta_k)&\to^p\bar{v}_n(\alpha^r+\zeta_k).
\end{aligned}$$

Because $\bar{v}_n$ is strictly increasing, $\bar{v}_n(\alpha^r-\zeta_k)<\bar{v}_n(\alpha^r)\leq r$ and $\bar{v}_n(\alpha^r+\zeta_k)>\bar{v}_n(\alpha^r)_+\geq r$. Because $\hat{\bar{v}}_n(\alpha)$ is an increasing function, this implies that $\hat{\bar{v}}_n^{-1}(r)\in(\alpha^r-\zeta_k,\alpha^r+\zeta_k)$ with probability approaching 1 for every k, which in turn implies $\hat{\bar{v}}_n^{-1}(r)\to^p\bar{v}_n^{-1}(r)$. By the continuous mapping theorem, $\left|\hat{\bar{v}}_n^{-1}(r)^{n-1}-\bar{v}_n^{-1}(r)^{n-1}\right|\to^p 0$. Therefore, for any $\alpha>0$, $\left|\hat{\bar{\sigma}}_n(\alpha,r)-\bar{\sigma}_n(\alpha,r)\right|\to^p 0$.

Next, notice that $\sup_{\alpha\in[0,1]}\hat{\bar{\sigma}}_n(\alpha,r)$ is bounded by a constant with probability approaching 1 as

$N \to \infty$. Moreover,

$$\mathbf{1}\left(r \leq \hat{\bar{v}}_n(\alpha)\right)\left(\hat{\bar{\sigma}}_n(\alpha, r) - c\right) \to^p \mathbf{1}\left(r \leq \bar{v}_n(\alpha)\right)\left(\bar{\sigma}_n(\alpha, r) - c\right)$$

for almost every α, because $\mathbf{1}\left(r \leq \hat{\bar{v}}_n(\alpha)\right)$ converges to $\mathbf{1}\left(r \leq \bar{v}_n(\alpha)\right)$ in probability and $\hat{\bar{\sigma}}_n(\alpha, r)$ converges to $\bar{v}_n(\alpha)$ in probability for all α such that $r \neq \bar{v}_n(\alpha)$. By the dominated convergence theorem,

$$\begin{aligned}\hat{\bar{\pi}}_n(r) - \Delta_n &= \int_0^1 \mathbf{1}\left(r \leq \hat{\bar{v}}_n(\alpha)\right)\left[\hat{\bar{\sigma}}_n(\alpha, r) - c\right] d\alpha^n + c \\ &\to^p \int_0^1 \mathbf{1}\left(r \leq \bar{v}_n(\alpha)\right)\left[\bar{\sigma}_n(\alpha, r) - c\right] d\alpha^n + c = \bar{\pi}_n(r)\end{aligned}$$

for any $r \geq 0$. To conclude the proof, we only need to show $\Delta_n \to^p 0$. To see this, just notice that $\hat{\bar{\pi}}_n(0) - \Delta_n \to^p \bar{\pi}_n(0)$ and $\hat{\underline{\pi}}_n(0) \to^p \underline{\pi}_n(0)$. By the definition of Δ_n, $\hat{\bar{\pi}}_n(0) = \hat{\underline{\pi}}_n(0)$, which implies that $\hat{\bar{\pi}}_n(0) \to^p \underline{\pi}_n(0) = \bar{\pi}_n(0)$. Therefore, $\Delta_n \to^p 0$. □

Proof of Proposition 12. We show that under these models bidders overbid compared to the RNBR, which implies Assumption 11.

Risk Aversion In a risk aversion model, bidder i best responds to the bid distribution. If her valuation is t, her first-order condition can be written as

$$t = b + \lambda_i^{-1}\left(\frac{1}{\sum_{j \neq i} \frac{g_{n,j}(b)}{G_{n,j}(b)}}\right) \leq b + \frac{1}{\sum_{j \neq i} \frac{g_{n,j}(b)}{G_{n,j}(b)}} = \rho_{n,i}^{-1}(b),$$

where $\lambda_i(x) = U_i(x)/U_i'(x)$ and U_i is the utility function of bidder i and satisfies $U_i(0) = 0$, $U_i'(x) > 0$ and $U_i''(x) \leq 0$. The inequality holds because $\lambda_i(0) = 0$ and $\lambda_i'(x) \geq 1$, which implies that $\lambda_i^{-1}(x) \leq x$.

Loser Regret Bidder i's first-order condition can be written as

$$t = b - h_i(t - b) + \frac{1}{\sum_{j \neq i} \frac{g_{n,j}(b)}{G_{n,j}(b)}} \leq b + \frac{1}{\sum_{j \neq i} \frac{g_{n,j}(b)}{G_{n,j}(b)}} = \rho_{n,i}^{-1}(b),$$

where h_i is the regret function. It satisfies $h_i(0) = 0$, $1 \geq h_i'(x) \geq 0$.

Choquet Expected Utility The first-order condition can be written as

$$t = b + \frac{\phi_i\left(\Pi_{j \neq i} G_j(b)\right)}{\phi_i'\left(\Pi_{j \neq i} G_j(b)\right)\Pi_{j \neq i} G_j(b) \sum_{j \neq i} \frac{g_j(b)}{G_j(b)}} \leq b + \frac{1}{\sum_{j \neq i} \frac{g_{n,j}(b)}{G_{n,j}(b)}} = \rho_{n,i}^{-1}(b),$$

where ϕ is a convex function that satisfies $\phi_i(0) = 0$ and $\phi_i'(x) > 0$. The inequality holds because $\phi_i(x)/x = (\phi_i(x) - \phi_i(0))/x \leq \phi_i'(x)$ by the property of convexity.

Loss Aversion Consider the setup studied in Lange and Ratan (2010) where a bidder gets monetary payoff from winning the auction. Then bidder i has a utility

$$U(x,e) = x - \eta_i \max\{e-x, 0\},$$

where x is the monetary profit, $\eta_i \in [0,1)$ captures the level of loss aversion and e is the reference point. We consider the case where e is determined by the expected payoff from bidding. If the bidder has valuation t and bids b, then $e = (t-b)\Pi_{j\neq i} G_j(b)$. Then bidder i's problem can be written as

$$\max_b (t-b)\left\{\Pi_{j\neq i} G_{n,j}(b) - \eta_i \Pi_{j\neq i} G_{n,j}(b)\left[1 - \Pi_{j\neq i} G_{n,j}(b)\right]\right\}$$

The first-order condition is

$$t = b + \frac{1-\eta_i + \eta_i \Pi_{j\neq i} G_{n,j}(b)}{1-\eta_i + 2\eta_i \Pi_{j\neq i} G_{n,j}(b)} \frac{1}{\sum_{j\neq i} \frac{g_{n,j}(b)}{G_{n,j}(b)}} \leq b + \frac{1}{\sum_{j\neq i} \frac{g_{n,j}(b)}{G_{n,j}(b)}} = \rho_{n,i}^{-1}(b).$$

$\square$