Two Results on Auctions with Endogenous Entry

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Abstract

First, the value distribution does not depend on the number of entrants under an independence assumption on entry signals. Second, separability of unobserved auction heterogeneity and private values carries over from potential bidders to entrants under several information structures.

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1 Introduction

Papers that study the identification or estimation of first-price auction models often place assumptions on the distribution of values among bidders who decided to enter the auction and whose bids are therefore observed. In many applications however it is plausible that prior to the bidding stage a set of potential bidders decides whether to enter the auction or not. The equilibrium of this entry game determines the set of entrants and the distribution of values among entrants is therefore an endogenous equilibrium object. This raises the question whether assumptions that are placed directly on the distribution of values among entrants are plausible if this distribution is determined by an entry game. In this paper we provide two results that support some of the assumptions that are commonly made in the literature.

We consider a very general entry framework where potential bidders receive a private signal before making their entry decisions. The framework nests commonly used entry models in auctions including the Samuelson entry model (Samuelson (1985)), the Levin-Smith entry model (Levin and Smith (1994)) and the Affiliated Signal entry model (Ye (2007), Gentry and Li (2014), Li, Lu, and Zhao (2015)). In our first result we show that the distribution of values conditional on the auction heterogeneity does not depend on the number of entrants if the values and the signals of any subset of potential bidders are independent of the entry signals of the other potential bidders. This result allows for interdependent values and under the independent private value paradigm, it is equivalent to requiring that signals and values are independent among potential bidders. This result suggests that a common assumption that is sometimes called "exogenous participation" can indeed be supported by a model with endogenous entry if the econometrician conditions on all auction characteristics. This is an important result because several papers in the literature use this assumption to achieve identification or for estimation purposes (Aradillas-López, Gandhi, and Quint (2013), Guerre, Perrigne, and Vuong (2009), Gillen (2009), Aryal, Grundl, Kim, and Zhu (2018), Grundl and Zhu (2019), Luo (2020), Jun and Zincenko (2022) and Haile, Hong, and Shum (2003)).[1]

[1] Liu and Luo (2017) propose a test for this assumption without conditioning on unobserved auction

To use this result it is crucial that the econometrician conditions on all auction characteristics that are observed by the bidders. In practice, some of these characteristics may not be observed by the econometrician and it is critical that one can condition on such unobserved heterogeneity. Under the independent private value setup, if the unobserved heterogeneity is not separable from the private values one can apply results from the non-classical measurement error literature (Hu and Schennach (2008), Hu, McAdams, and Shum (2013), Hu and Sasaki (2017)), but implementing this approach can be difficult. Therefore, much of the literature has considered the case where the unobserved heterogeneity and the private values of potential bidders are additively or multiplicatively separable. In our second result we focus on the setup with independent private values and show that such separability carries over from potential bidders to entrants if the potential bidders observe either a signal about their private value or the unobserved auction heterogeneity, but not if they observe both. This is an important result because separability of values and unobserved auction heterogeneity among entrants is a common assumption as it allows the use of deconvolution techniques (Kotlarski (1967), Evdokimov and White (2012), Li and Vuong (1998)) and thereby greatly simplifies identification and estimation (Krasnokutskaya (2011), Krasnokutskaya (2012), Li, Perrigne, and Vuong (2000), Grundl and Zhu (2019)).

Our first result is related to a strand of the structural auction literature that exploits exogenous variation in the number of bidders generated by an instrument such as a shifter of entry costs (e.g. Gentry and Li (2014) or Compiani, Haile, and Sant'Anna (2020)).[2] Our result is complementary to the instrumental variable approach, because some questions can be answered by exploiting variation in the number of entrants even if no instrument for entry is available, but for other questions the instrumental variable approach is necessary. Perhaps the most important object that cannot be identified with the help of our result is the distribution of values among potential bidders, because if entry is selective then the

heterogeneity.

[2]Haile and Kitamura (2019) is an excellent and recent survey discussing the different approaches to identification in the presence of unobserved heterogeneity in auctions, and also describe whether and how the different approaches allow for endogenous entry.

distribution of values among potential bidders differs from the distribution among entrants (Gentry, Li, and Lu (2017)). Moreover, because our result assumes that any subset of bidders' entry signals and values are independent of other bidders' entry signals, it should not be used in applications where values and entry signals may be dependent even after conditioning on the auction heterogeneity. However, if our assumption is plausible then one can exploit variation in the number of bidders without an instrument to identify or test for bidder risk aversion (e.g. Guerre, Perrigne, and Vuong (2009), Li, Lu, and Zhao (2015), Zincenko (2018), Grundl and Zhu (2019) Chen, Gentry, Li, and Lu (2020), Jun and Zincenko (2022)), identify ambiguity aversion or bidder beliefs (Aryal, Grundl, Kim, and Zhu (2018), Grundl and Zhu (2015)), or test for bidder collusion (Schurter (2020)).

2 Valuation Distribution Independent of n

There are N potential bidders who consider entering a first price auction. Potential bidders are indexed by $i \in \boldsymbol{N} = \{1, 2, \cdots, N\}$. To simplify the presentation, we assume that there are no auction specific characteristics.[3] Prior to bidding, a potential bidder i observes a private signal ξ_i (possibly multi-dimensional) about his value for the auctioned good, v_i. Potential bidder $i's$ entry strategy is a function $\phi_i : \xi_i \to \{0, 1\}$. He decides to enter if and only if ϕ_i takes the value 1.

Most commonly considered entry models fit into this framework for some ϕ_i. For example, in the Affiliated Signal entry model (Ye (2007), Gentry and Li (2014), Li, Lu, and Zhao (2015)), the entry strategy can be described by $\phi_i(\xi_i) = \mathbf{1}\{\xi_i > c_i\}$ where c_i is the cut-off that makes potential bidders indifferent between entering and not entering. In the Samuelson entry model (Samuelson (1985)), potential bidders know their values, i.e., $\xi_i = v_i$. In the Levin-Smith entry model (Levin and Smith (1994)), ξ_i is independent of v_i.

[3] If there are auction characteristics the results hold if we condition on them.

To simplify the notation, we suppress the argument of ϕ_i from now on. Let $\boldsymbol{v} = (v_1, v_2, \cdots, v_N)$,$\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_N)$, and $\boldsymbol{\phi} = (\phi_1, \phi_2, \cdots, \phi_N)$. Let $\mathbf{F}_{A,B|C}(\cdot, \cdot|\cdot)$ denote the joint distribution of A and B conditional on C. For example, the joint distribution of $\boldsymbol{v}$ conditional on $\boldsymbol{\phi}$ is $\mathbf{F}_{\boldsymbol{v}|\boldsymbol{\phi}}(\cdot|\cdot)$. Bidders are allowed to be asymmetric. If I is a subset of $\boldsymbol{N}$, let $\boldsymbol{v}_I = \{v_i\}_{i\in I}$, $\boldsymbol{\xi}_I = \{\xi_i\}_{i\in I}$, $\boldsymbol{\phi}_I = \{\phi_i\}_{i\in I}$ and $I^c = \{i \in \boldsymbol{N} : i \notin I\}$ be the complement of I. We write $\boldsymbol{\xi}_I = \mathbf{1}$ if $\xi_i = 1$ for all $i \in I$ and use subscript $-i$ to denote the set of potential bidders other than i. We next provide a general result which shows that if the value and the signal of each potential bidder are independent of the entry signals of the rivals, then the joint distribution of $\boldsymbol{v}_I$ conditional on X and $\boldsymbol{\phi}_I$ does not depend on $\boldsymbol{\phi}_{I^c}$.

Proposition 1. *If $(\boldsymbol{v}_I, \boldsymbol{\xi}_I) \perp \boldsymbol{\xi}_{I^c}$ for all $I \subset \boldsymbol{N}$, then $\mathbf{F}_{\boldsymbol{v}_I|\boldsymbol{\phi}_I,\boldsymbol{\phi}_{I^c}}(\cdot|\mathbf{1}, \cdot) = \mathbf{F}_{\boldsymbol{v}_I|\boldsymbol{\phi}_I}(\cdot|\mathbf{1})$ for all $I \subset \boldsymbol{N}$.*

Proof. The assumption implies that $(\boldsymbol{v}_I, \boldsymbol{\phi}_I) \perp \boldsymbol{\phi}_{I^c}$. Let $\Pr(\cdot|\cdot)$ denote the conditional probability. By conditional independence,

$$\mathbf{F}_{\boldsymbol{v}_I|\boldsymbol{\phi}_I,\boldsymbol{\phi}_{I^c}}(\cdot|\mathbf{1}, \cdot) = \frac{\mathbf{F}_{\boldsymbol{v}_I,\boldsymbol{\phi}_I|\boldsymbol{\phi}_{I^c}}(\cdot, \mathbf{1}|\cdot)\Pr(\boldsymbol{\phi}_{I^c} = \cdot)}{\Pr(\boldsymbol{\phi}_I = \mathbf{1}, \boldsymbol{\phi}_{I^c} = \cdot)} = \frac{\mathbf{F}_{\boldsymbol{v}_I,\boldsymbol{\phi}_I}(\cdot, \mathbf{1})\Pr(\boldsymbol{\phi}_{I^c} = \cdot)}{\Pr(\boldsymbol{\phi}_I = \mathbf{1})\Pr(\boldsymbol{\phi}_{I^c} = \cdot)} = \mathbf{F}_{\boldsymbol{v}_I|\boldsymbol{\phi}_I}(\cdot|\mathbf{1}),$$

where the second equality holds because $(\boldsymbol{v}_I, \boldsymbol{\phi}_I) \perp \boldsymbol{\phi}_{I^c}$. □

The key assumption of Proposition 1 is that the values and the signals of any subset of potential bidders are independent of the entry signals of the other potential bidders. In the independent private value setup, this assumption requires that both the signals and the values are independent across potential bidders. Notice that assuming independent signals alone is insufficient for this proposition. To see this, consider the case with two potential bidders. Bidder 1's value is $v_1 = v_1^1 + v_1^2$ where v_1^1 and v_1^2 are independent. Let $\xi_1 = v_1^1$ and $\xi_2 = v_1^2$ and suppose $\phi_2(\xi_2) = 1(\xi_2 > c)$ for some c that is in the interior of the support of v_1^2. Then the distribution of v_1 conditional on potential bidder 1 entering depends on potential bidder 2's entry decision.

Let $n = \sum_{i=1}^N \phi_i$ denote the number of entrants. Proposition 1 directly implies that

$\mathbf{F}_{\boldsymbol{v}_I|\boldsymbol{\phi}_I,n}(\cdot|\mathbf{1},\cdot) = \mathbf{F}_{\boldsymbol{v}_I|\boldsymbol{\phi}_I}(\cdot|\mathbf{1})$ for all $I \subset \{1,2,\cdots,N\}$, i.e., if all bidders in set I enter the auction, their joint distribution of values does not depend on the number of entrants. This implies that $\mathbf{F}_{v_i|\phi_i,n}(\cdot|1,\cdot) = \mathbf{F}_{v_i|\phi_i}(\cdot|\mathbf{1})$ for all $i \in \boldsymbol{N}$, which is a special case with $I = \{i\}$. If bidders are symmetric, let $F_{v|n}(\cdot|\cdot)$ be the value distribution of entrants conditional on n. Then $F_{v|n}(\cdot|\cdot) = \mathbf{F}_{v_i|\phi_i,n}(\cdot|1,\cdot) = \mathbf{F}_{v_i|\phi_i}(\cdot|\mathbf{1}) = F_v(\cdot)$, which is independent of n.

Corollary 1. *If $(\boldsymbol{v}_I, \boldsymbol{\xi}_I) \perp \boldsymbol{\xi}_{I^c}$ for all $I \subset \boldsymbol{N}$ and potential bidders are symmetric, then the marginal value distribution of entrants is independent of n.*

Many papers in the structural auction literature assume that the marginal value distribution of entrants is independent of n, which is often called "exogenous participation". Corollary 1 shows that this assumption holds under a fairly general entry model if any subset of potential bidders' values and signals are independent of the signals of the remaining potential bidders. In this case, the signals of other potential bidders shift the number of total entrants n "exogenously". Corollary 1 does not require values to be independent among the entrants and it can be easily generalized to the case where bidders are asymmetric and have different types.

3 Separability with Endogenous Entry

The discussion so far assumes that there are no auction characteristics. If there are auction characteristics, it is important that we can condition on all of them so that the results go through. In practice, there can be unobserved auction heterogeneity, i.e., some of the auction characteristics are not observed by the econometrician but observed by the bidders. To simplify the presentation, assume that the unobserved auction characteristics can be summarized by a one-dimensional variable and denote it by u. The standard approach to handle unobserved auction heterogeneity in the independent private value setup is to apply methods from the literature on non-classical measurement errors (Hu and Schennach (2008), Hu, McAdams, and Shum (2013), Hu and Sasaki (2017)). However, if the unobserved heterogeneity and private values among entrants are separable, one can apply deconvolution techniques to separate the effect of the unobserved heterogeneity. This setup greatly simplifies the analysis and is commonly used in the literature (Krasnokutskaya (2011), Krasnokutskaya (2012), Li, Perrigne, and Vuong (2000), Grundl and Zhu (2019)). It is based on the assumption that the value distribution of an *entrant* i satisfies $F_i(v|u) = F_i^*(v-u)$ or $F_i^*(v/u)$, or in other words, $v_i = u + v_i^*$ or $v = uv_i^*$ where v_i^* is entrant i's private valuation and follows the distribution F_i^* independent of u. In this section, we assume that v_i^* is independent of u and follows distribution F_i^* *before entry*. We then ask under what information structure this property carries over to the entrants. We focus on the independent private value setup where separability is most relevant but the results can be generalized to the case with interdependent values.

Proposition 2. *Suppose v_i^* and u are independent for potential bidder i; then v_i^* and u are independent conditional on potential bidder i entering the auction if one of the following conditions holds:*

(1). Potential bidder i's signal ξ_i is independent of v_i^* and u.

(2). Potential bidder i observes u and signal ξ_i, where (u, ξ_i) is independent of v_i^*.

(3). Potential bidder i's signal ξ_i is correlated with v_i^* and (v_i^*, ξ_i) is independent of u.

Proof. For Proposition 2(1), the entry strategy can be described by some function $\phi_i(\xi_i)$ that takes the value 0 or 1. Therefore, the joint distribution of (u, v_i^*) for entrant i is

$$\Pr(u \leq x, v_i^* \leq y | \phi_i(\xi_i) = 1) = \Pr(u \leq x, v_i^* \leq y) = \Pr(u \leq x) F_i^*(y).$$

For Proposition 2(2), the entry strategy can be described by $\phi_i(u,\xi_i)$. This is different from the formulation in the first case because u also enters ϕ_i. Then the joint distribution of (u, v_i^*) for entrant i is

$$\begin{aligned}
\Pr(u \le x, v_i^* \le y | \phi_i(u,\xi_i) = 1) &= E\{E[1(u \le x, v_i^* \le y)|u,\xi_i]|\phi(u,\xi_i) = 1\}. \\
&= E[1(u \le x) E[1(v_i^* \le y)|u,\xi_i]|\phi_i(u,\xi_i) = 1] \\
&= F_i^*(y)\Pr(u \le x|\phi_i(u,\xi_i) = 1).
\end{aligned}$$

The last step follows because v_i^* is independent of (u, ξ_i).

Lastly, following an analogous argument, one can show for Proposition 2(3) that

$$\Pr(u \le x, v_i^* \le y | \phi(\xi_i) = 1) = \Pr(v_i^* \le y|\phi(\xi_i) = 1)\Pr(u \le x).$$

$\square$

Under Proposition 2(1) and 2(2), v_i^* still follows the distribution F_i^*, i.e., there is no selection. Under Proposition 2(3), the distribution of v_i^* is different for potential bidders and for entrants because the signals lead to selection. Separability does not hold if potential bidders obtain some information about the private information v_i^* and about the unobserved auction heterogeneity u before they enter. In this case, v_i^* and u are no longer independent conditional on entering and we need to apply techniques from the non-classical measurement error literature.

4 Conclusion

This paper establishes two results for auctions with endogenous entry that support assumption which are commonly made in the structural auction literature.

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